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Gauge Construction and the Yang–Mills Heat Flow Over Real Four-Manifolds

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Chapter 1

Introduction

In 1954 C.N. Yang and R. Mills published a paper [59] which would give direction to the development of several areas of pure mathematics and theoretical physics over the coming decades. In an attempt to generalise the gauge theoretic framework of electromagnetism, Yang and Mills paved the way for the development of the standard model of particle physics, which is in fact a $SU(3)\times SU(2)\times U(1)$ gauge theory. It took several decades, however, for gauge theory to gain widespread mathematical interest, although since then it has had a profound impact on several areas of pure mathematics — most notably geometry, topology and non-linear partial differential equations (PDE). Of most interest to us is the interplay between Yang–Mills theory and the analysis of non-linear PDE. This thesis is devoted to exploring and understanding the fundamental contributions of Karen Uhlenbeck and Michael Struwe to the mathematical community's understanding of the analytic aspects of Yang–Mills theory.

Loosely speaking, a gauge is a way of giving a geometric object a 'co-ordinate system'. A gauge change can then be thought of simply as a change of 'co-ordinates', and a quantity is gauge invariant if it is preserved by gauge changes. Physical quantities are typically gauge invariant, and it is this gauge invariance which makes the analysis of PDE in gauge theory so difficult and interesting. In order to work with a model from a PDE perspective, one must usually *fix* a gauge, which is the same as stipulating a 'co-ordinate system'. Analogously to how choosing polar or cartesian co-ordinates may simplify a vector calculus problem in $\mathbb{R}^3$, one method of gauge fixing may yield a more tractable problem than another. As such, one of the underlying themes of this thesis is *finding good gauges*.

In Chapter 2 we develop some of the mathematical machinery necessary for this thesis and also recast the classical theory of electromagnetism as a gauge theory. The central mathematical objects which we work with are connections and curvature, denoted by D and F_D, respectively. The main problem of Yang–Mills theory is to find a connection which minimises the Yang–Mills functional,

$$\mathcal{YM}(D) = \int_M |F_D|^2 * (1).$$

From an analysis perspective, this was approached as a variational problem; however, compared with classical variational problems, this was made more delicate by the gauge invariance of the functional. This gave rise to two major obstacles:

- The classical notion of uniqueness was no longer valid.
- The functional was degenerate.

Moreover, the functional was required to be *coercive* in order for minimisers to be physically meaningful. To overcome these problems, it was clear that the gauge symmetry first needed to be broken. The major questions, then, were:

- Is it possible to break the gauge symmetry?
- If it is, is there a 'best' way to do this?

It is Uhlenbeck's 1982 paper 'Connections with L^p bounds on curvature' [53] which answers these two questions. Based on Uhlenbeck's result, the variational approach to Yang–Mills theory would ultimately prove successful in showing the existence of a minimising sequence of the Yang–Mills functional, although it was possible that this sequence necessarily converged over a different bundle. This is the main result of [46]. In Chapter 3 we review [53] and give an exposition of her argument and provide a discussion of its significance.

Dimension four is both physically and mathematically of significant interest in Yang–Mills theory. Physically, this is because one usually considers space-time as a four dimensional pseudo-Riemannian manifold. Mathematically, dimension four is critical in two senses. Geometrically, the functional is invariant under conformal transformations of the base manifold in dimension four. This is analogous to the case of the Dirichlet energy for harmonic maps in dimension two. Analytically, the relevant Sobolev embedding is critical when the minimal regularity assumptions are imposed on the space of connections. In a way which we will later make precise, this means that the coercivity condition is simplified, but that the elements of the gauge group are not necessarily continuous.

Suppose now that one has a minimiser of the Yang–Mills functional in the critical dimension with minimal regularity assumptions on the gauge group. This minimiser is then gauge equivalent to infinitely many other minimisers, but since elements of the gauge group are not necessarily continuous we have that a gauge change might *induce* a singularity in an otherwise smooth field. The question, then, is how to determine when singularities in Yang–Mills fields are inherent, and when they could be removed by a good choice of gauge. It is another of Uhlenbeck's papers, 'Removable Singularities in Yang–Mills Fields' [54] which not only answers this question in dimension four, but also provides a method to find a good gauge change. The entirety of Chapter 4 is devoted to an exposition of this paper in preparation for the Yang–Mills heat flow.

The heat flow method is well studied in geometric analysis, and it aims to construct a minimiser of a functional by deforming it along its lines of steepest descent. It was Atiyah and Bott in [1] who first suggested the heat flow method to explicitly construct a minimiser of the Yang–Mills functional, although it was Donaldson in [14] who first carried out this technique in the case of Kähler manifolds. This was a much celebrated result, although it still left open the main analytic problem of the Yang–Mills heat flow. This is because Donaldson's argument breaks down when one drops the regularity requirement of the gauge group. A much more subtle approach was required, and in a method strongly reminiscent of his earlier work on the harmonic map heat flow, [47] Michael Struwe demonstrated the short time existence to solutions of the Yang–Mills heat flow over real four-manifolds with minimal regularity assumptions in his 1994 paper 'The Yang–Mills Flow in Four Dimensions' [48]. The final chapter of this thesis is devoted to the analysis of Struwe's heat flow argument.

Chapter 2

Gauge Theory

2.1 Introduction

In this chapter we give a brief overview of the mathematics which is required for this thesis. Although to fully develop all of the machinery would take far too long - indeed, it would (and does) take many textbooks - we hope that this chapter will provide insight into some of the nuances of gauge theory. We first develop the theory of principal fibre bundles, which are central geometric objects in gauge theory. From these, we can develop the theory of vector bundles, which will be the main setting of our study in the subsequent chapters. We then give a brief exposition of the Yang–Mills functional, and then recast electromagnetism as a gauge theory. In this setting we consider both the static point electric charge, an electron, and also the magnetic analogue, a magnetic monopole. Finally, we bring several classical ideas of analysis into the language of gauge theory.

2.2 Principal Fibre Bundles

Although the main setting for our study in later chapters will be vector bundles, it is instructive and traditional to first develop the theory of principal bundles. In our case both the vector bundle and the principal bundle approach to gauge theory are equivalent, and we will explain this equivalence later. Unless stated otherwise, let M be a smooth, oriented, finite dimensional, compact and boundary free Riemannian manifold and G be a compact Lie group.

2.2.1 Definition and First Consequences

Definition 2.2.1 (Principal Fibre Bundle)**.** A principal bundle over M with structure group G is a triple (π, P, σ), where P is a manifold, called the total space, π is a smooth map

$$\pi : P \to M$$

and

$$\begin{aligned} \sigma : P \times G &\to G \\ (u, g) &\mapsto u \cdot g \end{aligned}$$

is a smooth right action of G on P such that the following conditions hold

(i) σ preserves fibres of P, i.e.

$$\pi(u \cdot g) = \pi(u) \tag{2.1}$$

for all $u \in P$ and $g \in G$.

(ii) (Local triviality) For each $x_0 \in M$ there exists a local trivialisation of P, (U, ψ), which consists of an open neighbourhood U of x_0 in M and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ of the form

$$\psi(u) = (\pi(u), \varphi(u)), \tag{2.2}$$

where $\varphi : \pi^{-1}(U) \to G$ satisfies

$$\varphi(u \cdot g) = \varphi(u)g \tag{2.3}$$

for all $u \in \pi^{-1}(U)$ and $g \in G$.

We call the manifold M the base space, and the spaces $\pi^{-1}(\pi(u))$ and $\pi^{-1}(x)$ for $u \in P,\ x \in M$ the fibre through u and the fibre over x, respectively.

If the action and projection map are understood we will often refer to P as a principal G-bundle and write $G \hookrightarrow P \to M$. By restricting the local trivialisation to a point we see that the fibres of P are diffeomorphic to G, although the standard fibres of G *are not Lie groups*. This is because there can be no identity element in the fibre. To see this, suppose $u \in \pi^{-1}(x)$. Then we must have

$$\varphi(u \cdot g) = \varphi(g) = \varphi(u)g,$$

which is not guaranteed to be true.

An important consequence of this definition is that we will be able to identify fibres of P as orbits of the group action. We prove this in the following lemma.

Lemma 2.2.2. *If (π, P, σ) is a principal fibre bundle with group G over M then each fibre through u for $u \in P$ is exactly the orbit of u under σ.*

Proof. We want to show that $\pi^{-1}(\pi(u)) = \{u \cdot g : g \in G\}$, and so a natural method is to show inclusions both ways. The inclusion $\pi^{-1}(\pi(u)) \supseteq \{u \cdot g : g \in G\}$ is immediate by (2.1). To see that $\pi^{-1}(\pi(u)) \subseteq \{u \cdot g : g \in G\}$ let $u' \in \pi^{-1}(\pi(u))$. We then aim to show that there necessarily exists a $g \in G$ such that $u' = u \cdot g$. Let (U, ψ) be a trivialisation as in (2.2). Then $\varphi(u), \varphi(u') \in G$, and so we let $g := (\varphi(u))^{-1}\varphi(u') \in G$. Then by (2.3) we have that $\varphi(u \cdot g) = \varphi(u')$. Therefore $\psi(u \cdot g) = (\pi(u \cdot g), \varphi(u \cdot g)) = (\pi(u'), \varphi(u')) = \psi(u')$. Then, since ψ is a diffeomorphism we have that $u \cdot g = u'$, and this concludes the proof. □

Remark 2.2.3. As noted in the introduction, a gauge can be thought of as giving a geometric object a 'co-ordinate system'. It is exactly the trivialisations of a principal fibre bundle which do this by providing diffeomorphisms which are compatable with the group action on fibres. Under this diffeomorphism, we may view a point in a fibre as corresponding to an element in the Lie group, although this identification is dependent on the trivialisation.

Consider now a fibre P_x, $u \in P_x$ and a trivialisation $\psi : P_x \to G$ such that $\psi(u) = (x, e)$. By the compatability condition of the trivialisations we can in some sense consider u to be a 'base point' of the fibre, and measure the 'location' of the other points of the fibre with respect to u and the group action.

Note that this 'location' is dependent on the trivialisation, since choosing a different trivialisation would specify a different 'base point'. Consequently, two points of the fibre would be related by a different element of the group action. For more on the geometric meaning of a gauge, the blog article of Terence Tao [49] is an excellent resource.

Example 2.2.4. Before we continue with our construction of principal bundles and their properties, it is useful to consider some concrete examples.

(i) As a first example of a principal fibre bundle, consider the case where P is a principal G-bundle and $P = M \times G$, i.e. P is globally trivialisable. We call such a bundle the *trivial bundle*.

(ii) The Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ is one of the earliest examples of a non-trivial principal fibre bundle. With the identification of S^3 as the unit sphere in $\mathbb{C}^2$, $g \in G = U(1)$ and $(z_1, z_2) \in S^3$. Note that $(z_1, z_2) \in S^3 \implies |z_1|^2 + |z_2|^2 = 1$. The action is defined as

$$\sigma((z_1, z_2), g) = (z_1 \cdot g, z_2 \cdot g) = (z_1 g, z_2 g).$$

This particular example will play a central role in our exposition of principal bundles, as it is of great physical significance. Although S^3 is embedded in 4-dimensional space, one can view S^3 as the compactification of $\mathbb{R}^3$, and as such get a geometric interpretation of what a non-trivial fibre bundle looks like. The image below was found at [25] and is useful in visualising the Hopf fibration. Each point on the sphere corresponds to a circle in $\mathbb{R}^3$ of the same colour, its 'fibre'. The Hopf fibration is well-studied, and we refer to, for example, Chapter 0.3 of [36], for a more in-depth discussion.

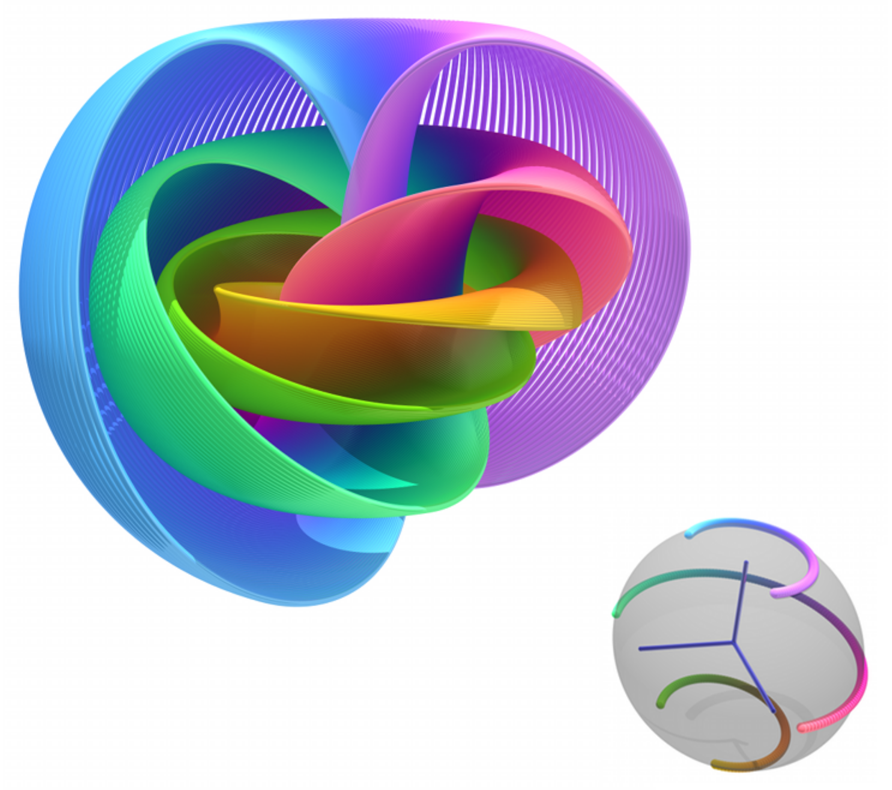

FIGURE 2.1: The Hopf fibration over S^2

(iii) Another principal bundle of central importance to our theory is the frame bundle of a vector bundle. For more on vector bundles, we refer to, for example, Chapter 2 of [27]. This example will motivate the discussion of the equivalence of the vector bundle and principal bundle approaches to gauge theory, and so we discuss it in more detail in Section 2.2.2.

In analogy with Riemannian geometry we must define transition maps between trivialisations on the intersection between two trivialisations. Suppose that P is a principal bundle over the base space M with structure group M, then if (U_α, ψ_α) and (U_β, ψ_β) are two trivialisations with non-trivial intersection, we want to establish some sort of relation

$$\psi_\beta \circ \psi_\alpha^{-1}(x, a) = (x, \psi_{\alpha\beta}(x)a)$$

for any $x \in M$ and $a \in G$. By various applications of the properties of a principal bundle, note that

$$\begin{aligned}
\psi_\beta \circ \psi_\alpha^{-1}(\pi(u), a) &= \psi_\beta \circ \psi_\alpha^{-1}(\pi(u), \varphi_\alpha(u)\varphi_\alpha^{-1}(u)a) \\
&= \psi_\beta \circ \psi_\alpha^{-1}(\pi(u), \varphi_\alpha(u \cdot \varphi_\alpha^{-1}(u)a)) \\
&= \psi_\beta \circ \psi_\alpha^{-1}(\pi(u \cdot \varphi_\alpha^{-1}(u)a)), \varphi_\alpha(u \cdot \varphi_\alpha^{-1}(u)a)) \\
&= \psi_\beta(u \cdot \varphi_\alpha^{-1}(u)a) \\
&= (\pi(u), \varphi_\beta(u \cdot \varphi_\alpha^{-1}(u)a)) \\
&= (\pi(u), \varphi_\beta(u)\varphi_\alpha^{-1}(u)a)).
\end{aligned}$$

Remark 2.2.5. Since G acts freely and transitively on fibres this definition does not depend on the choice of u.

This motivates the following definition:

Definition 2.2.6. Let $\{U_\alpha\}_{\alpha\in I}$, where I is an indexing set, be an open covering of M with diffeomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $u \mapsto (\pi(u), \varphi_\alpha(u))$. Then for any $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ we define a mapping

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$
$$\psi_{\alpha\beta}(\pi(u)) = \varphi_\alpha(u)\varphi_\beta^{-1}(u)$$

for any $u \in P$.

By this definition, we have

$$\begin{aligned} \psi_{\alpha\alpha} &= e \qquad \text{on } U_\alpha \\ \psi_{\alpha\beta}\psi_{\beta\alpha} &= e \qquad \text{on } U_\alpha \cap U_\beta \\ \psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} &= e \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

which are known as the *cocycle relations*. These transition functions are important in the theory of fibre bundles, and any fibre bundle (not necessarily principal) can actually be reconstructed from its transition functions - see, for instance, Chapter 5.3, Theorem 3.2 of [24].

Definition 2.2.7. Let $G \hookrightarrow P \to M$ be a principal G-bundle and let $\{U_\alpha\}$ cover M. Then $s_\alpha : U_\alpha \to P$ is called a *section* of P, or *local gauge*.

The relationship between sections of P and trivialisations of P is of fundamental importance to the theory of principal bundles. It is impossible to always define a global connection, and we find that local sections and local trivialisations of P are in a 1-1 correspondence.

Lemma 2.2.8. *Let (π, P, σ) be a principal G-bundle. Then there is a bijection between local sections and trivialisations of P.*

Proof. Let $\psi : \pi^{-1}(U) \to U \times G$ be a local trivialisation of P. We then define the *canonical section* as $s_U(x) = \psi^{-1}(x, e)$, where the subscript U denotes the coordinate patch. Conversely, given a section $s : U \to P$, we are able to define a trivialisation of P by taking advantage of Lemma 2.2.2. By definition, we have that

$$\pi^{-1}(U) = \bigsqcup_{x\in U} \pi^{-1}(x) = \bigsqcup_{x\in U} \{s(x) \cdot g : \ g \in G\},$$

and we claim that $\psi : \pi^{-1}(U) \to U \times G$ given by $\psi(s(x) \cdot g) = (x, g)$ defines a local trivialisation (ψ, U). By Lemma 2.2.2, we have that ψ is a bijection and

$$\psi(s(x) \cdot g) = (\pi(s(x)), \varphi(s(x) \cdot g)),$$

where $\varphi(s(x) \cdot g) = g$. Therefore

$$\varphi(s(x) \cdot g \cdot g') = gg' = \varphi(s(x) \cdot g)g' \implies \varphi(s(x) \cdot g \cdot g') = \varphi(s(x) \cdot g)g'$$

for all $x \in U$ and $g, g' \in G$. The smoothness of $\psi^{-1}(x,g) = s(x) \cdot g$ follows from the smoothness of the action, and the smoothness of ψ will follow from the smoothness of $\varphi : \pi^{-1}(U) \to G$, and so to see the smoothness of φ, choose a trivialisation of P, (U, ψ'), where $\psi'(p) = (\pi(p), \varphi'(p))$ and $\varphi'(p) = (s(x) \cdot g) = g$. Now, we have that

$$\varphi'(p) = \varphi'((s \circ \pi)(p) \cdot g) = ((\varphi \circ s \circ \pi)(p))g,$$

and so

$$g = \varphi'(p)((\varphi \circ s \circ \pi)(p))^{-1} = \varphi(p),$$

and so we see that φ is the composition of smooth functions, and so is therefore smooth. Therefore (U, ψ) is a local trivialisation of P. □

We have the following important consequence:

Corollary 2.2.9. *A principal fibre G-bundle (π, P, σ) is trivial if and only if it admits a global section.*

Proof. If P admits a global section, then this induces a trivialisation (M, ψ) such that $\psi : \pi^{-1}(M) \to M \times G$, and so P is trivial. Conversely, if P is trivial, then the canonical section associated to this trivialisation is globally defined. □

Definition 2.2.10. Let $G \hookrightarrow P \to M$ be a principal G-bundle, and let $\Phi : \pi^{-1}(U) \to \pi^{-1}(U)$ be a bundle automorphism such that

$$\Phi(p \cdot g) = \Phi(p) \cdot g \tag{2.4}$$

for all $p \in P$ and $g \in G$. We call such an automorphism a *local gauge transformation*.

Note that condition (2.4) means that the bundle automorphism is a fibrewise automorphism, i.e., that $\pi \circ \Phi = \pi$. Since compositions and inverses of automorphisms are also automorphisms, we have the following definition:

Definition 2.2.11. Let $G \hookrightarrow P \to M$ be a principal G-bundle. The collection of all gauge transformations of P form a group, called the *gauge group* of P and we denote this $\mathcal{G}$.

Remark 2.2.12. The terminology introduced above is motivated by physics, and it should be noted that in the physics literature it is common to refer to both the structure group of a principal bundle, and the group of fibre-preserving automorphisms of the bundle as the gauge group.

2.2.2 The Frame Bundle

One of the most important and most-studied examples of a principal fibre bundle is the frame bundle. The frame bundle should be taken as a motivating example of a principal fibre bundle, and in this section we will review the construction of a frame bundle, although its importance will only become clear in Section 2.6. These objects are well studied, and we refer to, for example, [36] Chapter 3.3, or [29], Chapter 10.11 for a detailed treatment.

Informally, a frame of a finite dimensional vector space is an ordered basis. When dealing with vector bundles, one requires a way of associating a frame to each fibre. More concretely, let M be a smooth d-dimensional manifold (not necessarily oriented or Riemannian) and E a vector bundle of rank n over M. A *frame* of E at $x \in M$ is an ordered basis $p = (v_1, ..., v_n)$ for E_x. Any such frame p gives rise to a linear isomorphism

$$\tilde{p} : \mathbb{R}^n \to E_x$$
$$\tilde{p}(e_i) = v_i,$$

where $\{e_i\}_{i=1,...,n}$ is the usual basis of $\mathbb{R}^n$. For each $x \in M$ we denote by $F(E_x)$ the set of all frames at $x \in M$. We then define the frame bundle to be

$$F(E) = \bigsqcup_{x \in M} F(E_x).$$

For each $p \in F(E_x)$, we define a surjective map

$$\pi : F(E) \to M$$
$$\pi(p) = x.$$

We define $\sigma : F(E) \times GL(n, \mathbb{R}) \to F(E)$ as follows: For each $(p, g) \in F(E) \times GL(n, \mathbb{R})$ with $p = (p_1, ..., p_n) \in F(E_x)$ and $g = (g^i_j) \in GL(n, \mathbb{R})$, we let $\sigma(p, g) = p \cdot g$ be the frame $(\hat{p}_1, ..., \hat{p}_n)$ at $x \in M$ defined by

$$\hat{p}_j = p_i g^i_j, \qquad \text{for } j = 1, ..., n.$$

This is just matrix multiplication from the right, which makes it clear that σ satisfies the requirements of a group action, namely $p \cdot (g_1 g_2) = (p \cdot g_1) \cdot g_2$ for $g_1, g_2 \in GL(n, \mathbb{R})$, and also that $p \cdot e = p$. Moreover, by this definition we also have $\pi(p \cdot g) = \pi(p)$ for all $p \in F(E)$, $g \in GL(n, \mathbb{R})$. In general, when we have that the group action is given by matrix multiplication from the right, we will write $\sigma(p, g) := R_g(p) = pg$. To check that this defines not only a smooth manifold structure, but also a bundle structure is detailed calculation. Such a calculation is out of scope of this chapter, and so we refer to [36], Chapter 3.3 for a construction of the frame bundle associated to a tangent bundle of a manifold, or to, for example, [29], Chapter 10.11 for a more general construction.

A local section of the frame bundle $s : U \to F(E)$ assigns to each $x \in M$ a frame $s(x) = (u_1(x), ..., u_n(x))$ at x and is called a *frame field* or *moving frame*. We will be able to say more about the frame bundle of (E, π, M), a vector bundle of rank n, if we know more about M. Namely, we have the following Theorem:

Theorem 2.2.13. *Each vector bundle (E, π, M) of rank n of an oriented Riemannian manifold M with a bundle metric has structure group $G \subseteq SO(n)$.*

Proof. Let (f, U) be a bundle chart for E,

$$f : \pi^{-1}(U) \to U \times \mathbb{R}^n.$$

Let $\{e_i\}_{i=1,...,n}$ be the canonical basis of $\mathbb{R}^n$, and let $v_1, ..., v_n$ be the sections of $\pi^{-1}(U)$ with $f(v_i) = e_i$ for each $i = 1, ..., n$. By applying Gram-Schmidt orthogonalisation (see, for example [16], Chapter 2.1d) to the sections $v_1(x), ..., v_n(x)$ for each $x \in U$ we obtain sections $w_1, ..., w_n$ of $\pi^{-1}(U)$ for which $w_1(x), ..., w_n(x)$ are an orthonormal basis with respect to the bundle metric for each $x \in U$. By

$$\begin{aligned} f' : \pi^{-1}(U) &\to U \times \mathbb{R}^n \\ \lambda^i w_i(x) &\mapsto (x, \lambda^1, ..., \lambda^n) \end{aligned}$$

we get a bundle chart which maps the basis $w_1(x), ..., w_n(x)$ for each $x \in U$ onto a Euclidean orthonormal basis of $\mathbb{R}^n$. By applying this orthonormalisation procedure for each bundle chart we obtain a new bundle atlas whose transition maps always map a Euclidean orthonormal basis of $\mathbb{R}^n$ into another orthonormal basis of $\mathbb{R}^n$. Since the manifold is oriented, the transition maps of the atlas all have positive functional determinant, and with this we see that the Jacobian matrix between transition maps for the vector bundle all have determinant 1. Therefore, the largest group which the transition maps take their values in is $SO(n)$. □

With this, we may say that the frame bundle of an a vector bundle is a principal G bundle, where $G \subseteq SO(n)$, if we have that the base space is an oriented Riemannian manifold. In the subsequent chapters it is frequently assumed that the structure group of the vector bundle being cosidered is a subgroup of $SO(n)$, and this is the reason why this is valid - we are always working with oriented Riemannian manifolds.

2.3 Connections of a Fibre Bundle

We now come to one of the most important ideas in this chapter - the connection. In order to be able to 'differentiate' sections of the bundle we will need a method of identifying points in adjacent fibres so that the definition of a derivative makes sense. There are a number of equivalent ways to view a connection, and the theory can be independently developed exclusively for vector bundles, which is the approach commonly taken in analysis literature - see, for example, Chapter 4 of [27]. Before giving definitions, we first discuss the problem which a connection is introduced to solve and hopefully motivate the necessity and importance of connections purely from a geometric point of view. Although we will spend the majority of this section constructing connections on principal fibre bundles, the motivation is the same for any fibre bundle, and so we introduce some general notions first before specialising our attention.

Let E be a fibre bundle with standard fibre F, $\gamma : [-1, 1] \to M$ a path in M such that $\gamma(0) = x$ and $\gamma'(0) = X \in T_x M$ and let $s : M \to E$ be a section of E. Then we would like to define a differential $ds(x) : T_x \to T_{s(x)} E$. With Riemannian geometry in mind, we define

$$ds(x)X = \frac{d}{dt} s(\gamma(t))\Big|_{t=0} = \lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}.$$

Now, if $E = M \times F$ has a trivial bundle structure, then we may consider a section as a map $s : M \to F$, and the above definition makes sense. Unfortunately, in most cases of interest to us, the bundle structure is not trival, and we have that $s(\gamma(t))$ and $s(\gamma(0))$ are in *different spaces*, and there is no canonical identification between them. It is clear then, that the above definition doesn't make sense, and that it must be altered in order to recover a meaningful definition.

Although a canonical identification may not be possible, there should be a way to identify adjacent fibres, seeing as they are all diffeomorphic to F, after all. We would like to 'connect' adjacent fibres in a meaningful way, and this leads to the notion of *parallel transport*. Just as in Riemannian geometry, parallel transport leads to the notion of a *covariant derivative*, where it is covariant in the sense that the derivative is independent of the co-ordinate chart chosen.

Suppose then that we have a family of diffeomorphisms (in a sense which have yet to make clear) for the above path $\gamma(t) \in M$

$$P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}.$$

These are far from unique, as we will see, and this is one of the reasons why the theory of connections is so interesting. Although highly non-unique, these isomorphisms are not arbitrary. Suppose that we have the same setting as above, then we define the *covariant derivative* of s along γ to be

$$\nabla_X s(x) := \frac{d}{dt}\Big[(P_\gamma^t)^{-1} \circ s(\gamma(t))\Big]\bigg|_{t=0}.$$

Such an identification of nearby fibres defines a path in the fibre $E_{\gamma(0)}$, which means that

$$\nabla s(x) : T_x M \to T_{s(x)} F,$$

where $T_{s(x)}F$ is the tangent plane of the fibre. Note that if E is a vector bundle of rank n, then $T_{s(x)}F \simeq \mathbb{R}^n$, and if E is a principal G-bundle, then $T_{s(x)}F \simeq \mathfrak{g}$.

2.3.1 The Horizontal Subbundle

If one considers the fibre bundle E as a manifold, then there is a well defined tangent space $T_u E$ for each $u \in E$. We may then construct the tangent bundle of E, denoted TE. Inside this bundle there is a distinguished subbundle, called the *vertical subbundle* which consists of all vectors in TE which are tangent to any fibre.

Definition 2.3.1. Let (E, M, π, F) be a vector bundle. The *vertical bundle* $VE \to E$ is the subbundle of $TE \to E$ defined by

$$VE := \{\xi \in TE \big|\ \pi_* \xi = 0\},$$

where $\pi_* : TE \to TM$ is the pushforward of π.

This subbundle is uniquely determined by the structure of the fibre bundle. With this definition, we see that the covariant derivative is a map

$$\nabla s(x) : T_xM \to V_{s(x)}E.$$

We require that

(i) $\nabla_X s(x)$ is independent of the choice of γ, so long as $\gamma'(0) = X$.

(ii) The map $\nabla s(x) : T_xM \to V_{s(x)}E$ is linear.

To continue with our theory of connections we require the definition of a *horizontal subbundle*. Unlike the vertical subbundle, the horizontal subbundle is non-unique, and it is the assignation of a horizontal subbundle which determines the isomorphisms P^t_γ in the definition of the covariant derivative. We will explore the construction of horizontal subbundles more for principal bundles, although the definition is valid for all fibre bundles.

Definition 2.3.2. Let (E, π, M, F) be a fibre bundle. A *horizontal subbundle* is a smooth distribution HE on the total space TE such that

$$TE = VE \oplus HE$$

and

$$VE \cap HE = \{0\}$$

With this we come to our first definition of a connection. This definition is valid for any fibre bundle, although the additional structure of principal and vector bundles will impose additional conditions on a connection.

Definition 2.3.3. For any $X \in TE$, we will write $X = vX_v + hX_h$, where $X_v \in VE$ and $X_h \in HE$. A *connection* on the fibre bundle (E, π, M, F) is a smooth distribution HE on the total space such that $HE \oplus VE = TE$, $V_uE \cap H_uE = \{0\}$. For each $u \in E$, the fibre $H_uE \subset T_uE$ is called the *horizontal subspace* at u.

Although it is true that this does in fact define a connection, it is in no way clear how, or that it should be possible to, at least locally, define such horizontal subspaces. The above definition leads to the most general notion of a connection possible. Due to the generality of a fibre bundle, it is quite technical to show how this choice of horizontal distribution determines the diffeomorphisms P^t_γ, and so we will now restrict our attention to the case where E is a principal fibre budle. For more about this general notion of connection we refer to, for example, [34], Chapter 17.

2.3.2 Connections on a Principal Fibre Bundle

The theory of connections on prinicipal fibre bundles is well developed, and most of what is introduced here will be taken from Chapter 6 of [35] and Chapter II of [28].

As a first step towards explaining how a horizontal distribution defines a family of diffeomorphisms for a principal fibre bundle, we introduce the following definition:

Definition 2.3.4. Let (P, π, M) be a principal G-bundle on which a smooth horizontal distribution has been chosen, $\gamma : [0,1] \to M$ a path through M and $\tilde{\gamma} : [0,1] \to P$ a path through P such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [0,1]$. Then we call $\tilde{\gamma}(t)$ a *horizontal* path through P if $\frac{d}{dt}\tilde{\gamma}(t) \in H_{\tilde{\gamma}(t)}E$.

The existence of such horizontal paths is crucial to defining the family of diffeomorphisms. Fortunately, the following lemma guarantees the existence of a horizontal path given a path in M. The proof is rather long, and so we cite the the textbook in which it was found, rather than reproducing it here.

Lemma 2.3.5 (Proposition 3.1 of [28])**.** *Let (P, π, M) be a principal G-bundle, $\gamma : [0,1] \to M$ a path in M and $u_0 \in P$. Then, there is a unique horizontal lift $\tilde{\gamma} : [0,1] \to P$ such that $\tilde{\gamma}(0) = u_0$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and $\tilde{\gamma}(t)$ is a horizontal path.*

This allows us to define the diffeomorphisms P^t_γ as follows: Let $\gamma : [0,1] \to M$ be a smooth curve in M and let $u_0 \in \pi^{-1}(\gamma(0))$ be arbitrary. Let $\tilde{\gamma} : [0,1] \to E$ be the unique horizontal lift of $\gamma(t)$ with $\tilde{\gamma}(0) = u_0$. We then define $P^t_\gamma(u_0) = \tilde{\gamma}(t) \in \pi^{-1}(\gamma(t))$. This map is a diffeomorphism since $R_a \circ P^t_\gamma = P^t_\gamma \circ R_a$ for each $a \in G$ and G acts freely and transitively on fibres. Note that because of the extra conditions imposed a principal fibre bundle, this family of diffeomorphisms is not quite enough to define a connection on a principal fibre bundle. Additionally, we require that $P^t_\gamma : P_{\gamma(0)} \to P_{\gamma(t)}$ be G-equivariant. This means that for any $u \in P$ and $g \in G$, we have

$$P^t_\gamma(ug) = P^t_\gamma(u)g.$$

Remark 2.3.6. It should be reiterated that this construction is dependent on the choice of the horizontal subspace. Since there are in general infinitely many choices, it is not always clear as to what constitutes a 'good' choice of horizontal subbundle, although we will further develop this notion later.

Definition 2.3.7 ([28], page 12)**.** Let M be a smooth manifold. Then, a *1-parameter group of diffeomorphisms* is a smooth map $\varphi : \mathbb{R} \times M \to M$ such that

1. For each $t \in \mathbb{R}$, the map $\varphi_t := \varphi(t, \cdot) : M \to M$ is a diffeomorphism.
2. For all $t, s \in \mathbb{R}$ and $u \in M$ there holds $\varphi_{t+s}(u) = \varphi_t(\varphi_s(u))$.

Each 1-parameter group *induces* a vector field $X \in \Gamma(TM)$ as follows: For each $x \in M$, X_x is the tangent vector to the curve $t \mapsto \gamma_t(x)$ at $x = \varphi_0(x)$. Conversely, each $X \in \Gamma(TM)$ *generates* a local 1-parameter group of diffeomorphisms, although this is the content of Proposition 1.5 of [28] and we refer there for the precise statement and proof.

Now, let $x \in M$ and $\alpha : [0,1] \to M$ be such that $\alpha(0) = x$ and $\alpha'(0) = X$. Then by Lemma 2.3.5 we have that there exists a horizontal path $\tilde{\alpha} : [0,1] \to P$ such that $\tilde{\alpha}(0) = u$ and $\tilde{\alpha}'(0) \in H_uP$. We then define

$$\begin{aligned} \mathrm{Hor}_u : T_xM &\to T_uE \\ \alpha'(0) &\mapsto \tilde{\alpha}'(0). \end{aligned}$$

This map is well defined and independent of the choice of α, and this follows from Lemma 2.3.5. Moreover, by the uniqueness of the horizontal lift, it is injective. Suppose then, that for a path $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = X \in T_xM$, and let $u \in P_{\pi(u)}$. Then

$$\begin{aligned}
\mathrm{Hor}_{ug}(X) &= \frac{d}{dt} P^t_\gamma(ug)\big|_{t=0} \\
&= \frac{d}{dt} R_g \circ P^t_\gamma(u)\big|_{t=0} \\
&= (R_g)_* \frac{d}{dt} P^t_\gamma(u)\big|_{t=0} \\
&= (R_g)_* \mathrm{Hor}_u(X).
\end{aligned}$$

With this, we then have formulated the extra condition which a connection on a principal bundle must satisfy.

Definition 2.3.8. A connection on a principal fibre bundle $G \hookrightarrow P \to M$ is a smooth distribution HP on TP such that $HP \oplus VP = TP$, $HP \cap VP = \{0\}$ and

$$H_{ug}P = (R_g)_* H_u P.$$

Remark 2.3.9. Note the similarity between the definition of a connection on a principal fibre bundle and a connection on a fibre bundle. This shouldn't be surprising, as a principal fibre bundle is first and foremost a fibre bundle. It is the extra structure which turns it into a *principal* fibre bundle which necessitates the extra condition on the definition of a connection, as we saw above.

We now describe a way of explicitly constructing this horizontal subspace. This will lead us to our next definition of a connection on a principal fibre bundle, and we will show that it is equivalent to our first definition.

Definition 2.3.10. Let $G \hookrightarrow P \to M$ be a principal G-bundle over M and $\mathfrak{g}$ the Lie algebra of G. Let $A \in \mathfrak{g}$ and $t \mapsto \exp(tA)$ a path in G. Then, the 1-parameter group of diffeomorphisms

$$\begin{aligned}
\varphi : \mathbb{R} \times P &\to P \\
(t, u) &\mapsto u \cdot \exp(tA)
\end{aligned}$$

for any $u \in P$ induces a vector field $A^\#$ on P. We call $A^\#$ the *fundamental vector field* corresponding to A.

A first consequence of this definition is that since the action of G on P preserves fibres, $A^\#_u$ is always vertical for any $u \in P$, and in fact *any* $v \in V_uP$ is such that $v = A^\#(u)$ for some $A \in \mathfrak{g}$.

Definition 2.3.11. Let $G \hookrightarrow P \to M$ be a principal fibre bundle. A connection on P is a smooth $\mathfrak{g}$-valued 1-form A which satisfies the following two conditions:

(i) $(R_g)^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ for all $g \in G$,

(ii) $\omega(A^\#) = A$ for all $A \in \mathfrak{g}$.

We will now show how this definition uniquely determines a horizontal subbundle of TP, and therefore diffeomorphisms with which we can define the covariant derivative. We claim that by defining

$$H_uP := \{v \in T_uP : \ \omega_p(v) = 0\}$$

we recover the first definition of a connection on a principal fibre bundle. Firstly, note that $H_uP \cap V_uP = \{0\}$ for any $u \in P$, since if there exists non-trivial $v \in H_uP \cap V_uP$, then $v = A^{\#}(u)$ for some $A \in \mathfrak{g}$ *and* $\omega_u(v) = 0 = \omega_u(A^{\#}(u)) = \omega$, and so $v = 0$. To show that that $H_uP \oplus V_uP = T_uP$, it suffices to show that $\dim H_uP + \dim V_uP = \dim T_uP$, which follows from the rank-nullity theorem if one considers the surjective map $\pi_* : T_uP \to T_{\pi(u)}M$ with kernel V_uP.

To show that this definition implies that $H_{ug}P = (R_g)_* H_uP$, note that condition (i) implies that for $v \in T_{ug^{-1}}P$,

$$A_u((R_g)_{*ug^{-1}}(v)) = \mathrm{Ad}_{g^{-1}}\omega_{ug^{-1}}(v).$$

Therefore, if $v \in H_uP$, then

$$\begin{aligned}\omega_{ug}((R_g)_{*u}(v)) &= \omega_{ug}((R_g)_{*ugg^{-1}}(v)) \\ &= \mathrm{Ad}_{g^{-1}}(\omega_u(v)) \\ &= 0,\end{aligned}$$

and so $H_uP \subseteq (R_g)_* H_{ug}P$. Next, suppose that $w \in H_{ug}P$. Since $(R_g)_{*u}$ is an isomorphism, there exists $v \in H_uP$ such that $w = (R_g)_{*u}(v)$, and we must show that $v \in H_uP$. Note, however, that $\omega_u(v) = \omega_u((R_{g^{-1}})_{*ug}(w)) = \mathrm{Ad}_g(\omega_{ug}(w)) = 0$. Therefore the reverse inclusion is proven, and we have that Definition 2.3.11 implies Definition 2.3.8. To see that Definition 2.3.8 implies Definition 2.3.11, one must simply reverse the above arguments. The diagramme below represents the G-equivariance of the horizontal subbundle diagrammatically and is taken from [37], Chapter 10.

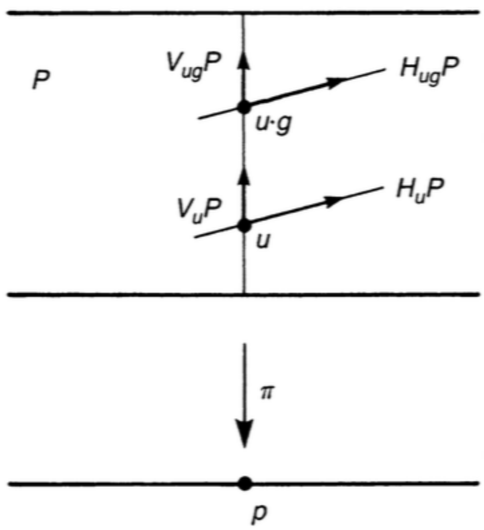

FIGURE 2.2: The horizontal subspace is preserved by the pushforward of the right action

Before moving on, we briefly pause to introduce the notion of the space of horizontal p-forms. These are a generalisation of the form ω in the sense that ω is a horizontal Ad-equivariant 1-form.

Definition 2.3.12. Let $G \hookrightarrow P \to M$ be a principal fibre bundle. Then we define

$$\Omega^p_{\text{Ad}}(P; \mathfrak{g})$$

to be the space of all horizontal $\mathfrak{g}$-valued, Ad-equivariant p-forms.

Now, given ω a $\mathfrak{g}$-valued one form on P, one may define the *exterior covariant derivative* to be

$$d_\omega : \Omega^p_{\text{Ad}}(P; \mathfrak{g}) \to \Omega^{p+1}_{\text{Ad}}(P; \mathfrak{g})$$

by having $d\eta$ act only on horizontal vectors for $\eta \in \Omega^p_{\text{hor}}(P; \mathfrak{g})$, ie

$$d_\omega \eta(X_1, ..., X_{p+1}) = d\eta(hX_1, ..., hX_{p+1}),$$

where hX_i is the horizontal component of X_i as determined by ω. For X, Y smooth vector fields on M, we define the *curvature* of ω to be

$$\Omega := d_\omega \omega.$$

Note that

$$\begin{aligned} d_\omega \omega(X, Y) &= d\omega(hX_h, hY_h) \\ &= (hX_h)\omega(hY_h) - (hY_h)\omega(hX_h) - \omega([hX_h, hY_h]) \\ &= \omega([hX_h, hY_h]), \end{aligned}$$

and so we see that $\Omega(X, Y) = 0$ if and only if $[hX_h, hY_h] \in HE$, and so one can interpret the curvature as a measure of the lack of integrability (in the sense of Frobenius) of the horizontal subbundle determined by ω. See, for example, [31] Chapter 1.4, for more on integrability in the sense of Frobenius.

Theorem 2.3.13 (Cartan Structure equation)**.** *Let $G \hookrightarrow P \to M$ be a smooth principal G-bundle with connection form ω and let Ω be the curvature of ω. Then*

$$\Omega = d\omega + \omega \wedge \omega.$$

Proof. By linearity, we must check the following three cases:

(i) v and w are vertical vectors,

(ii) v is vertical and w is horizontal,

(iii) v and w are both horizontal.

In case (i) and (ii) we have that $\Omega(v, w) = 0$, so we must show that $d\omega(v, w) + \omega \wedge \omega(v, w) = 0$. In the case that v and w are both vertical, we have that $d\omega(v, w) = v\omega(w) - w\omega(v) - \omega([v, w])$. Recall that since v, w are both vertical, they are the fundamental vector field of some elements $A, B \in \mathfrak{g}$, so we have that $v = A^\#$, $w = B^\#$, and so $\omega_u(A^\#(u)) = A, \omega_u(B^\#(u)) = B$ for all u in $P_{\pi(u)}$. Therefore $A^\#(\omega(B^\#)) = B^\#(\omega(A^\#)) = 0$, since tangent vectors annihilate constant functions. Therefore, we have that $\omega([v,w]) = \omega \wedge \omega(v,w) = [\omega(v), \omega(w)]$, which is true, although is non-trivial to show - see, for example, Proposition 5.8.8 of [36].

If w is horizontal and v is vertical, then we have that $w = B^{\#}$ for some $B \in \mathfrak{g}$, as before. Furthermore, we have that $\omega \wedge \omega(v,w) = 0$, since ω is a horizontal form. We must show, then, that $d\omega(v,w) = 0$. By definition, we have that $d\omega(v,w) = v\big(\omega(w)\big) - w\big(\omega(v)\big) - \omega([v,w])$. Moreover, since $\omega(B^{\#})$ is a constant map and $\omega(v) = 0$ since v is horizontal, the proof boils down to showing that $\omega([v,w]) = 0$. We must show, then, that $[v,w]$ is horizontal. Since $w = B^{\#}$ is a fundamental vector field, it is induced by a 1-parameter family of diffeomorphisms $b(t) = \exp(tB)$, we have that

$$[v,w] = \lim_{t \to 0} \frac{R_{b(t)*}w - w}{t},$$

and since w is horizontal, we have that $R_{b(t)*}w$ is horizontal by Definition 2.3.8, and so it follows that $[v,w]$ is horizontal, and so $\omega([v,w]) = 0$.

In the last case, we have that v, w are both horizontal, and the equality follows immediately from the definition of Ω. □

Remark 2.3.14. Throughout the literature on the subject of gauge theory and principal bundles various authors adopt various conventions about how to denote the the wedge product of Lie algebra valued forms where the bilinear map used to construct the wedge product is the Lie bracket. All of these are equivalent, and this can be seen if written out in local co-ordinates. For example, other common notation denotes the wedge product of Lie algebra valued differential forms as $\frac{1}{2}[\omega,\omega] = \frac{1}{2}[\omega_i,\omega_j]dx^i \wedge dx^j$, and we see that this is in fact equivalent to our notation since $\omega \wedge \omega = \omega_i\omega_j dx^i \wedge dx^j = \frac{1}{2}[\omega_i,\omega_j]dx^i \wedge dx^j$, where $\omega_i, \omega_j \in \mathfrak{g}$. Other common notation includes $[\omega \wedge \omega]$ and $\frac{1}{2}[\omega \wedge \omega]$. It should be noted that these different notations, although equivalent, will differ by when a factor of $\frac{1}{2}$ is included. For this reason, one method of notation should be adopted and adhered to, so as to avoid erroneous multiplicative factors.

Perhaps unsurprisingly, we have that the curvature satisfies:

$$R_g^*\Omega(X,Y) = \mathrm{Ad}_{g^{-1}} \circ \Omega(X,Y),$$

where $g \in G$ and $X, Y \in \Gamma(TM)$. To see this, note that the exterior covariant derivative commutes with pull-backs - see, for example, Chapter 5.7 of [36]. We have then that

$$\begin{aligned} R_g^*\Omega(X,Y) &= R_g^* d\omega(hX,hY) \\ &= d\big(R_g^*\omega\big)(hX,hY) \\ &= d\big(\mathrm{Ad}_{g^{-1}} \circ \omega\big)(hX,hY). \end{aligned}$$

Now, Ad is a linear transformation and so commutes with d, and so we have that

$$R_g^*\Omega(X,Y) = \mathrm{Ad}_{g^{-1}} \circ \Omega(X,Y).$$

2.4 Associated Bundles

The associated bundle is a crucial link between the theory of vector and principal fibre bundles. Principal and vector bundles represent different settings physically, although the same information is encoded in each. As in the case with connections, the study of associated bundles can be constructed very generally, and we present the general theory initially before specialising our attention to the case which interests us the most - the case of associated vector bundles.

Let $G \hookrightarrow P \to M$ be a principal fibre bundle with action $\sigma : P \times G \to P$ given by $\sigma(u, g) = u \cdot g$ and let F be a smooth manifold on which G acts smoothly on the left. We define an action on $P \times F$ given by

$$(P \times F) \times G \to P \times F$$
$$((u, \xi), g) \mapsto (u \cdot g, g^{-1} \cdot \xi)$$

for all $g \in G$ and $f \in F$. We then define an equivalence relation as follows: $(u_1, \xi_1) \sim (u_2, \xi_2)$ if and only if there exists a $g \in G$ such that $(u_2, \xi_2) = (u_1 \cdot g, g^{-1} \cdot \xi_1)$. We denote the equivalence class $[u, v] := \{(u \cdot g, g^{-1} \cdot \xi) : \ g \in G\}$. We then denote $P \times_G F$ the orbit space of $P \times_G F$ modulo this action, and we equipt $P \times_G F$ with the quotient topology. We then define a mapping $\pi_G : P \times_G F \to M$ by $\pi_G([u, \xi]) = \pi(u)$. This map is well defined, since $\pi(u \cdot g) = \pi(u)$. Now, to check that $P \times_G F$ has the local trivialisation properties required of a fibre bundle, let U be an open subset in M and $\psi : \pi^{-1}(U) \to U \times G$, and let $s : U \to \pi^{-1}(U)$ be the canonical section. We then define a map $\tilde{\Phi} : U \times F \to \pi_G^{-1}(U)$ by

$$\tilde{\Phi}(x, \xi) = [s(x), \xi].$$

Note that $\pi_G(\tilde{\Phi}(x, \xi)) = \pi_G([s(x), \xi]) = \pi(s(x)) = x$, and so $\tilde{\Phi}$ does in fact map into $\pi_G^{-1}(U)$. Note that since $[u \cdot g, \xi] = [u, g \cdot \xi]$ we get that for a fixed point $u \in \pi^{-1}(x)$, $\pi_G^{-1}(x) = \{[u, \xi] : \ \xi \in F\}$. Therefore

$$\pi_G^{-1}(U) = \bigsqcup_{x \in U} \pi_G^{-1}(x).$$

One can check that this map diffeomorphism, although we refer to page 381 of [36], for example, for a proof. Now, it is actually the inverse of this map which is the trivialisation we are after. Define

$$\tilde{\Psi} : \pi_G^{-1}(U) \to U \times F$$
$$[s(x), \xi] \mapsto (x, \xi).$$

Likewise, this map is a diffeomorphism. Therefore $(P \times_G F, M, \pi_G, F)$ is a locally trivial bundle. Now, let U_α and U_β be two open sets in M with nonemphy intersection and corresponding trivialisations $\tilde{\Psi}_\alpha, \tilde{\Psi}_\beta$, respectively.

Then

$$\begin{aligned}\tilde{\Psi}_\alpha \circ \tilde{\Psi}_\beta(x,\xi) &= \tilde{\Psi}_\alpha([s_\beta(x),\xi]) \\ &= \tilde{\Psi}_\alpha([s_\alpha(x)\cdot\psi_{\alpha\beta}(x),\xi]) \\ &= \tilde{\Psi}_\alpha([s_\alpha(x),\psi_{\alpha\beta}(x)\cdot\xi]) \\ &= (x,\psi_{\alpha\beta}(x)\cdot\xi)\end{aligned}$$

which is smooth, since the transition functions on P and the left action on F are smooth. Therefore $\tilde{\Psi}_\alpha, \tilde{\Psi}_\beta$ has the structure of a smooth fibre bundle, and we call it the fibre bundle associated with P given by the action of G on F.

Example 2.4.1. Throughout this thesis there will be several associated bundles of major importance.

(i) The main setting for our analysis in subsequent chapters will be vector bundles. Given a principal bundle, a vector space and a smooth left action of the structure group on this vector space, one can construct an associated vector bundle as above where $F \simeq \mathcal{V}$, where $\mathcal{V}$ is a finite dimensional vector space. In this case, we may define a representation $\rho : G \to GL(\mathcal{V})$, and we denote the total space of the corresponding associated bundle by $P \times_\rho \mathcal{V}$. In particular, let E be a vector bundle of rank n with structure group $G \subseteq SO(n)$ with corresponding frame bundle $G \hookrightarrow F(E) \to M$. We then have that there is a canonical isomorphism $F(E) \times_\rho \mathcal{V} \simeq E$ if ρ is the identity representation. This isomorphism should not be surprising, since both bundles have the same trivialisations, although we refer to, for example, Chapter 18.3 of [34] or Example 1 on page 124 of [51] for an explicit proof. In such a way, we may consider a vector bundle as being a bundle associated to its frame bundle.

(ii) Another associated bundle of central importance is the *automorphism bundle* Aut $P = P \times_G G$, where the action is the Adjoint action of G on G. To motivate the name, consider the following: Let the smooth right action of G on G in the definition of the principal fibre bundle be multiplication from the right, and recall that a local gauge is given by $s : U \to P$ and that a gauge transform is a bundle automorphism $\Phi : \pi^{-1}(U) \to \pi^{-1}(U)$ such that $\Phi(pg) = \Phi(p)g$ for $p \in P$ and $g \in G$. Note that each bundle automorphism defines a map $\varphi : P \to G$ such that $\Phi(s(x)) = s(x)\varphi(s(x))$ for all $x \in U$. Therefore, we see that $\varphi(s(x)g) = g^{-1}\varphi(s(x))g$, and this defines a section

$$\begin{aligned}S : U &\to \text{Aut } P \\ x &\mapsto [p,\varphi(p)]\end{aligned}$$

where $p \in \pi^{-1}(x)$ is arbitrary. Conversely, any section of Aut P defines an Ad-equivariant function φ, and thus a bundle automorphism of P. Therefore, it is often convenient to think of a gauge transformation as a section of Aut P. Since the composition and inverse of an automorphism is still an automorphism, we find that the elements of Aut P form a group under composition, which we call the *gauge group*, and denote $\mathcal{G}$. Note that this definition agrees with Definition 2.2.11.

(iii) The last associated bundle of major importance in the following theory is the *adjoint bundle* ad $P = P \times_{\mathrm{Ad}} \mathfrak{g}$. This is the vector bundle associated with P with standard fibre $\mathfrak{g}$ (as a vector space) given by the Adjoint representation on $\mathfrak{g}$

$$\begin{aligned}\mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g})\\ g &\mapsto \mathrm{Ad}_g\end{aligned}$$

where

$$\begin{aligned}\mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g}\\ A &\mapsto gAg^{-1}.\end{aligned}$$

We will show shortly that connections on P can be considered as a section of $\Omega^1(\mathrm{ad}\, P)$.

Given an associated vector bundle, we would like a way to consider sections of it. Let $G \hookrightarrow P \to M$ be a principal G-bundle and $P \times_\rho \mathcal{V}$ the vector bundle of rank n associated to P by the left G-action on $\mathcal{V}$. Assume that the action of G on P is given by multiplication on the right. For an open $U \subseteq M$, a map $\phi : \pi^{-1}(U) \to \mathcal{V}$ is said to be *ρ-equivariant* if

$$\phi(ug) = \rho(g^{-1})\phi(u)$$

for all $u \in \pi^{-1}(U)$ and $g \in G$. Note that this generalises the Ad-equivariance of ω in Definition 2.3.11 (i). With such a map we may construct a local section of $s_\phi : U \to P \times_\rho \mathcal{V}$ as follows: Let $x \in U$ and select an arbitrary $u \in \pi^{-1}(x)$. The pair $(u, \phi(u))$ dertermines a point $[u, \phi(u)] \in P \times_\rho \mathcal{V}$, and this point is independent of the choice of $u \in \pi^{-1}(x)$ since G acts freely on the fibres of P and by the equivariance of the map. Therefore, the map

$$s_\phi(x) = [u, \phi(u)]$$

for *any* $u \in \pi^{-1}(x)$ is well defined. By varying $x \in U$, we see that this defines a smooth section $s_\phi : U \to \pi_\rho^{-1}(U)$. Conversely, given a smooth section $s : U \to \pi_\rho^{-1}(U)$, we define a map $\phi_s : \pi^{-1}(U) \to \mathbb{R}^n$ as follows: Let $u \in \pi^{-1}(U)$, then $x = \pi(u) \in U$, so $s(x) \in \pi_\rho^{-1}(x)$. There is then a unique element, which we call $\phi_s(u) \in \mathbb{R}^n$, such that $s(x) = [u, \phi_s(u)]$. Then, for any $u \in \pi^{-1}(U)$ and any $g \in G$ we have we define $\phi_s(ug)$ as follows: Since $\pi(ug) = \pi(u) = x$, $\phi_s(ug)$ is the unique element of $\mathbb{R}^n$ such that $s(x) = [ug, \phi_s(ug)]$. From the definition of the quotient map, however, we have that $s(x) = [u, \phi_s(u)] = [ug, \rho(g^{-1})\phi_s(u)]$, which implies that $\rho(g^{-1})\phi_s(u) = \phi_s(ug) = \rho(g^{-1})\phi_s(u)$, and so $\phi_s : \pi^{-1}(U) \to \mathcal{V}$ is an equivariant map. Furthermore, $s \mapsto s_\phi$ and $\phi \mapsto \phi_s$ are inverses of eachother, and we see that there is a bijection between equivariant $\mathcal{V}$-valued maps of P and sections of $P \times_\rho \mathcal{V}$.

We may in fact go further than concluding a bijection between the space of ρ-equivariant horizontal $\mathcal{V}$-valued maps of P and the sections of $P \times_\rho \mathcal{V}$ and assert that there exists a canonical isomorphism

$$\Omega^k_\rho(P; \mathcal{V}) \simeq \Omega^k(P \times_\rho \mathcal{V}),$$

where by $\Omega^k_\rho(P;\mathcal{V})$ we mean the space of ρ-equivariant horizontal $\mathcal{V}$-valued p-forms. Although the isomorphism is canonical, we refer to the literature for a proof - for example Chapter 19.4 of [34]. Since $\omega \in \Omega^1_{\text{Ad}}(P;\mathfrak{g})$, we may consider it as a section of $\Omega^1(\text{ad } P)$ under the canonical isomorphism. With this isomrphism we will be able to give an explicit formulation of the exterior covariant derivative of bundle-valued differential forms. Towards such ends, we have the following theorem, although we refer to, for example, Theorem 4.5.6 of [35] for a proof.

Theorem 2.4.2. *Let $G \hookrightarrow P \to M$ be a smooth principal fibre bundle with connection form ω, $\mathcal{V}$ a finite dimensional vector space, $\rho : G \to GL(\mathcal{V})$ a representation of G on $\mathcal{V}$ and $\varphi \in \Omega^k_\rho(P;\mathcal{V})$, a horizontal, ρ-equivariant k-form on P. Then*

$$d_\omega\varphi = d\varphi + d\rho \wedge \varphi.$$

Here, and in the following, $d\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ is the differential of the representation defined as

$$d\rho(A) = \frac{d}{dt}\rho(\exp(tA))\Big|_{t=0}$$

In particular, we have $\varphi \in \text{ad } P$ that

$$d_\omega\varphi = d\varphi + [\omega, \varphi],$$

since the differential of the Adjoint action is the adjoint action.

Remark 2.4.3. There appears to be a lack of consistency in our definition of the wedge product of Lie algebra valued forms and what have written for $d_\omega\varphi$ for $\varphi \in \text{ad } P$. Note, however, that by $[\omega, \varphi]$, we mean that

$$[\omega, \varphi] = \omega \wedge \varphi - \varphi \wedge \omega,$$

and so there is no inconsistency in our notation.

We may now prove the *first Bianchi identity*.

Theorem 2.4.4 (First Bianchi identity)**.** *Let $G \hookrightarrow P \to M$ be a smooth principal G-bundle with connection ω and curvature $\Omega = d_\omega\omega$. Then*

$$d_\omega\Omega = 0.$$

Proof. We compute directly that

$$\begin{aligned} d_\omega\Omega &= d\Omega + [\omega, \Omega] \\ &= dd\omega + d\omega \wedge \omega - \omega \wedge d\omega + [\omega, d\omega] + [\omega, \omega \wedge \omega] \\ &= [\omega, \omega \wedge \omega] \\ &= [\omega_i dx^i, \omega_j\omega_k dx^j \wedge dx^k] \\ &= \omega_i\omega_j\omega_k(dx^i \wedge dx^j \wedge dx^k - dx^j \wedge dx^k \wedge dx^i) \\ &= 0 \end{aligned}$$

□

Consider now a vector bundle $E := P \times_\rho \mathcal{V}$ associated to P where ρ is the identity representation on E. Let

$$D : \Gamma(E) \to \Omega^1(E)$$

be a connection on E. We may equivalently consider

$$D \in \Omega^1(E) \otimes (\Omega^0(E))^* = \Omega^1(\text{End } E),$$

where $\Omega^1(\text{End } E)$ is the bundle of fibre-preserving endomorphisms of E. To see that G acts on End E by the Adjoint action, we refer to Section 2.5.2. We have then, that End E is a vector bundle with standard fibre $\mathfrak{gl}(\mathcal{V})$ on which G acts by the Adjoint action, and so ad $P \subseteq$ End E. Moreover, each connection on E can be written in local co-ordinates as

$$D = d + A,$$

where $A \in \mathfrak{gl}(\mathcal{V})$. For a proof of this, see, for example, Chapter 4 of [27]. For a connection induced by ω, we see that $D = d + s^*\omega$, where $s : U \to P$ is a local gauge. Therefore, for a connection on E induced by a connection on P, we find that in local co-ordinates we can write

$$D = d + A =: d_A,$$

where $A \in \mathfrak{g}$. Similarly, for $\varphi \in \Gamma(E)$, we identify $d_A\varphi$ with $d_\omega\varphi'$, where $\varphi' \in \Omega^0_\rho(P; \mathcal{V})$so that

$$d_A\varphi = d\varphi + A \wedge \varphi.$$

Induced connections on vector bundles have the following properties:

- The connection induced on $\Omega^0(E)$ is called the *covariant derivative*, is denoted ∇ and is a map

  $$\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M),$$

  where $\nabla\sigma(V) =: \nabla_V\sigma$ for $\sigma \in \Gamma(E)$, $V \in T_xM$ with the properties:

  - ∇ is tensorial in V:

    $$\nabla_{V+W}\sigma = \nabla_V\sigma + \nabla_W\sigma \qquad \text{for } V, W \in TM,\ \sigma \in \Gamma(E)$$
    $$\nabla_{fV}\sigma = f\nabla_V\sigma \qquad \text{for } f \in C^\infty(M, \mathbb{R}),\ V \in \Gamma(TM).$$

  - ∇ is $\mathbb{R}$-linear in σ:

    $$\nabla_V(\sigma + \tau) = \nabla_V\sigma + \nabla_V\tau \qquad \text{for } V \in T_xM,\ \sigma, \tau \in \Gamma(E).$$

  - ∇ satisfies the Liebniz Rule:

    $$\nabla_V(f\sigma) = V(f) \cdot \sigma + f\nabla_V\sigma \qquad \text{for } f \in C^\infty(M, \mathbb{R}).$$

  - We may locally identify $\nabla = d + A =: \nabla_A$, where $A \in \Gamma(\mathfrak{g} \otimes T^*M)$

- They are *metric*, which means that for all $\mu, \nu \in \Omega^0(E)$, we have

$$d\langle \mu, \nu \rangle = \langle \nabla \mu, \nu \rangle + \langle \mu, \nabla \nu \rangle,$$

  where $\langle \cdot, \cdot \rangle$ is a bundle metric, see, for example, Chapter 2 of [27].

- The induced connection on $\Omega^p(E)$ is called the *exterior covariant derivative*, is denoted d_A and has the following properties:

  - $d_A = \nabla_A$ on $\Omega^0(E)$
  - $d_A(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge (d_A \theta)$ for $\omega \in \Omega^p(M)$ and $\theta \in \Omega^q(E)$
  - The L^2 dual to the operator d_A is defined as $d_A^* : \Omega^{k+1}(E) \to \Omega^k(E)$, where

$$(d_A \mu, \nu) = (\mu, d_A^* \nu),$$

    for any $\mu \in \Omega^k(E)$, $\nu \in \Omega^{k+1}(E)$, where $(\cdot, \cdot) = \frac{1}{2} \int_M \langle \cdot, \cdot \rangle * (1)$ and $* : \Omega^k(E) \to \Omega^{n-k}(E)$ is the Hodge star operator - see, for example, Chapter 22.6 of [21].

- The Hodge Laplacian $\Delta_A := d_A^* d_A + d_A d_A^*$ and the rough Laplacian $\nabla_A^* \nabla_A$ are associated by the Weitzenböck identity: For $\phi \in \Omega^k(E)$, we have

$$\Delta_A \phi = \nabla_A^* \nabla_A \phi + \mathrm{Rm} \# \phi + F_A \# \phi, \tag{2.5}$$

  where $\#$ is any multilinear map with smooth co-efficients and Rm is the Riemannian curvature. This can be written out exactly, but we will not need this, and so we refer to Theorem 4.3.3 of [27], among other places, for the exact formulation.

- The *curvature* of an induced connection d_A is given by $F_A := d_A \circ d_A$, and locally can be written as

$$F_A = dA + A \wedge A,$$

  When written out in local co-ordinates, we have

$$(F_{ij}) = \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j] \right) dx^i \wedge dx^j.$$

- The curvature of an induced connection also satisfies the first and second Bianchi identities:

$$d_A F_A = 0$$
$$d_A^* d_A^* F_A = 0.$$

  To show that $d_A F_A = 0$ is almost identical to Theorem 2.4.4, and so we refer to, for example, Theorem 4.1.1 of [27] for a proof. That $d_A^* d_A^* F_A = 0$ is called the second Bianchi identity, and can be shown by noting that for $\phi \in \Omega^0(\text{ad } P)$, we have

$$\begin{aligned}(d_A^* d_A^* F_A, \phi) &= (F_A, d_A d_A \phi), \\ &= (F_A, [F_A, \phi]) \\ &= 0,\end{aligned}$$

where $(\cdot, \cdot)$ is the inner product defined in section 2.7 and the last step follows by the cyclic property of the trace inner product.

2.5 Transformation Laws

Since objects in differential geometry are in general only described locally, we describe briefly how connections and curvature transform under gauge changes and changes of co-ordinates. In this section we briefly describe the transformation behaviour of objects in both the principal, and vector bundle setting.

2.5.1 Principal Bundles

We briefly describe here the transformation laws associated witht he local descriptions of objects on principal bundles. We present this primarily for completeness, since it will be the transformation laws of vector bundles which we will use in the following chapters. Everything in this subsection is common in literature, and can be found at, for example, Chaters 5 and 6 of [36] and Chapter 2 of [35].

On any principal bundle, there exists a distinguished Lie-algebra valued 1-form called the *Cartan 1-form*, denoted by θ. This is the one-form which sends each left invariant vector field on G to its generator at T_eG. Namely, we have that for each $g \in G$, $\theta(g) = \theta_g : T_g(G) \to \mathfrak{g}$ is given by

$$\theta_g(v) = (L_{g^{-1}})_{*g}(v)$$

for each $v \in T_g(G)$. Now, let (U_α, ψ_α) a trivialisation of P such that $U_\alpha \cap U_\beta \neq \emptyset$. Recall now the canonical section $s_\alpha(x) := \psi_\alpha^{-1}(x, e)$. We then define a *local connection form* by

$$\omega_\alpha := s_\alpha^* \omega,$$

and this completely determines ω on $\pi^{-1}(U_\alpha)$. This is known in the physics literature as the *local gauge potential*. We also define the *local curvature form* as

$$\Omega_\alpha := s_\alpha^* \Omega,$$

and this is known in the physics literature as the *local field strength*. In local co-ordinates, we have that the local curvature form is given by

$$\Omega_\alpha := d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha,$$

which is perhaps unsurprising. For a proof, we refer to, for example, page 350 of [36]. Now, suppose that we have two trivialisations $(U_\alpha, \psi_\alpha),\ (U_\beta, \psi_\beta)$

such that $U_\alpha \cap U_\beta \neq \emptyset$. To each trivialisation we have the canonical cross sections $s_\alpha, s_\beta \in \Gamma(P)$ and the local curvature forms $\omega_\alpha, \omega_\beta \in \Omega^1(M; \mathfrak{g})$. We also have the transition function (also known as a local gauge change) $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. The transformation law is then given by:

$$\omega_\beta = \psi_{\beta\alpha}\omega_\alpha\psi_{\alpha\beta} + \psi^*_{\alpha\beta}\theta,$$

and one can show, although we refer to, for example, page 305 of [36] for a proof, that this is equivalent to

$$\omega_\beta = \psi_{\beta\alpha}\omega_\alpha\psi_{\alpha\beta} + \psi_{\beta\alpha}d\psi_{\alpha\beta}. \tag{2.6}$$

To motivate the transformation law for a canonical section of a principal bundle and recall the transition law, Definition 2.2.6, and note the following:

$$\begin{aligned} s_\beta(x) &= \psi_\beta^{-1}(x, e) \\ &= \psi_\alpha^{-1}(x, \psi_{\alpha\beta}(x)) \\ &= s_\alpha(x)\psi_{\alpha\beta}(x). \end{aligned}$$

Now, let $E := P \times_\rho \mathcal{V}$ be a vector bundle associated to P by a representation. Then suppose that $\sigma_\alpha : U_\alpha \to E$ and $\sigma_\beta : U_\beta \to E$ are sections of E such that $U_\alpha \cap U_\beta \neq \emptyset$. The corresponding transformation law on is given by:

$$\sigma_\beta = \rho(\psi_{\alpha\beta}^{-1})\sigma_\alpha. \tag{2.7}$$

Therefore, since we may view the curvature as a section of $\Omega^2(\text{ad } P)$, we have that it transforms as

$$\Omega_\beta = \psi_{\beta\alpha}\Omega_\alpha\psi_{\alpha\beta}. \tag{2.8}$$

2.5.2 Vector Bundles

As always, let M be a smooth, finite dimensional, compact, orientable, boundary-free Riemannian manifold. Let P be a principal bundle and $E := P \times_\rho \mathcal{V}$, where $\rho : G \to GL(\mathcal{V})$ is a representation on $\mathcal{V}$, a finite dimensional vector space. Firstly, consider the case where $\mathcal{V} = \mathbb{R}^n$ and ρ is the identity representation. Suppose we have a cover $\{U_\alpha\}$ and $\psi\alpha\beta : U_\alpha \cap U_\beta \to G$, where $G \subseteq SO(n, \mathbb{R})$ is the structure group of the bundle. Let $\mu_\alpha, \mu_\beta \in \Gamma(E)$ defined on U_α and U_β, respectively. Recall that by equation (2.7) we must have

$$\begin{aligned} \mu_\beta &= \rho(\psi_{\alpha\beta}^{-1})\mu_\alpha \\ &= \psi_{\beta\alpha}\mu_\alpha. \end{aligned}$$

When one considers sections of $\Omega^1(E)$, we must similarly have

$$\begin{aligned} \psi_{\beta\alpha}(d + A_\alpha)\mu_\alpha &= (d + A_\beta)\mu_\beta \\ \psi_{\beta\alpha}(d + A_\alpha)\psi_{\alpha\beta}\mu_\beta &= (d + A_\beta)\mu_\beta \\ (d + \psi_{\beta\alpha}d\psi_{\alpha\beta} + \psi_{\beta\alpha}A_\alpha\psi_{\alpha\beta})\mu_\beta &= (d + A_\beta)\mu_\beta. \end{aligned}$$

Therefore, we get that $A_\beta = \psi_{\beta\alpha}d\psi_{\alpha\beta} + \psi_{\beta\alpha}A_\alpha\psi_{\alpha\beta}$ describes the *transformation behaviour* of the connection forms. Note that d_A does not transform as a tensor, but the difference of two connections does. The space of all connections on a given vector bundle E is therefore an *affine space* with underlying space $\Omega^1(\text{ad } P)$. Similarly, if one considers the curvature as a section of $\Omega^2(\text{ad } P)$, we must have

$$F_\beta = \psi_{\beta\alpha} F_\alpha \psi_{\alpha\beta},$$

and so in contrast to connection forms, the curvature transforms as a tensor.

Now, let $S \in \mathcal{G} = \text{Aut } P$, and $d_A \in \Omega^1(\text{ad } P)$, then, we have that S acts on d_A by conjugation. Therefore $S^*(d_A) = S^{-1} \circ d_A \circ S$. If $\sigma \in \Gamma(E)$, then we have

$$\begin{aligned} S^*(d_A)\sigma &= S^{-1} \circ d_A(S\sigma) \\ &= S^{-1}(d + A)(S\sigma) \\ &= d\sigma + S^{-1}(dS)\sigma + S^{-1}AS\sigma. \end{aligned}$$

Therefore, we can see that $S^*(A) = S^{-1}dS + S^{-1}AS$ describes the transformation behaviour of connections under gauge changes. Similarly, since $F_A \in \Omega^2(\text{ad } P)$, we have that

$$S^* F_A = S^{-1} \circ F_A \circ S.$$

Similarly, if $S_\alpha, S_\beta \in \mathcal{G}$ are two local gauge transformations, on the overlap $U_\alpha \cap U_\beta$, we have that

$$S_\beta = \psi_{\beta\alpha} S_\alpha \psi_{\alpha\beta}$$

2.6 Equivalence of the Vector Bundle Approach to Gauge Theory

In this section we give a brief explanation of the equivalence of our construction and the approach where vector bundles are exclusively considered. Recall from section 2.2.2 that, given a vector bundle of rank n, E, there exists a distinguished principal bundle $F(E)$, called the frame bundle. We saw in example 2.4.1 (i) that each vector bundle is canonically isomorphic to the vector bundle associated to $F(E)$ via the identity representation.

In Section 2.4 we stated the properties of connections on vector bundles which are induced by connections on principal bundles. For the vector and principal bundle approaches to be equivalent it must be true that *every* connection on a vector bundle with these properties arises as an induced connection on its frame bundle. For the case of vector bundles this is in fact true that every metric connection on a vector bundle is induced from a connection on its frame bundle, although we refer to, for example, Chapter 11.9 of [34] for the proof. Specifically for vector bundles, a connection is metric if and only if its connection form takes its values in $\mathfrak{g}$, the Lie algebra of the structure group G of the vector bundle, although we refer to, for example, Proposition 3.22 of [58] for a proof. We have then, that a connection on a vector bundle is induced by a connection on its frame bundle if and only if the corresponding connection forms in compatible trivialisations take their values in $\mathfrak{g}$. It should be reiterated that this is in general not the case, but in most applications this equivalence will be taken advantage of without specific mention to any theorems.

Many sources do away with reference to the frame bundle all together and refer to the endomorphism bundle End E, which is the bundle of endomorphisms of E which fixes the base manifold and preserves fibres. We have the canonical isomorphism:

$$\text{End } E \simeq \text{ad } F(E).$$

We refer to, for example, Example 4.9 of [3] for a proof, although we note that this isomorphism should not be difficult to motivate, seeing as both bundles have the same trivialisations. It is common in literature to refer to this as ad E or Ad E in order to make contact with the theory of principal bundles.

It should not be surprising since E is associated to $F(E)$ by the identity representation that we have the canonical isomorphism

$$\mathcal{G} := \text{Aut } F(E) \simeq \text{Aut } E := \bigsqcup_{x \in M} \text{Aut}_x \, E$$

where Aut E is the bundle automorphisms which fix the base manifold and preserves fibres. Again, this follows from the fact that both bundles have the same trivialisations, although we note that a proof appears as Proposition 2 of Part 1.1 in [39], amongst other places.

We have then, that one may equivalently consider vector bundles and work with exactly the same information as in the principal bundle case. Namely, if we have a vector bundle E over M, then we have:

- The gauge group $\mathcal{G} := \text{Aut } E$,
- The endomorphism bundle ad E,
- Every metric connection can be considered as an element of $\Omega^1(\text{ad } E)$, and the curvature as a section of $\Omega^2(\text{ad } E)$.
- The structure group of E acts on $\mathcal{G}$ by conjugation,
- $\mathcal{G}$ acts on ad E by the Adjoint action.
- With respect to the inner product which we will introduce in section 2.7, we have that $|\Omega|^2 = |F_A|^2$, although we refer to Corollary 19.16 of [34], for example, for a proof.

This structure is induced from the principal bundle structure of the frame bundle, although it is common to make no reference to the frame bundle and only consider the vector bundle. Although vector and principal bundles represent different structures in physcis, we see that mathematically they contain the same information, and it constitutes no loss of generality to pass back and forth between the approaches as is convenient.

2.7 The Yang–Mills Functional

Let M be a smooth, compact, finite dimensional, oriented, boundary-free Riemannian manifold of dimension m. We discussed previously that it is equivalent to consider a vector bundle E over M or its associated frame bundle $F(E)$, and so in keeping with the notation of the literature which we will later analyse we will adopt the vector bundle approach.

Let E be a smooth vector bundle of rank n over M with G-structure. We are under the standing assumption that G and $\mathfrak{g}$ are compact, and so we have the natural trace inner product on $\mathfrak{g}$ given by the negative of the Killing form, i.e.

$$A \cdot B = -\operatorname{tr}(AB)$$

for $A, B \in \mathfrak{g}$. With this and the inner product on p-forms, we may introduce a pointwise scalar product for $\mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2 \in \mathrm{ad}_x \otimes \wedge^p T_x^* M$ by

$$\langle \mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2 \rangle := \mu_1 \cdot \mu_2 \langle \omega_1, \omega_2 \rangle,$$

where we define

$$\langle \omega_1, \omega_2 \rangle = *(\omega_1 \wedge *\omega_2).$$

This yields by linear extension an L^2 inner product on $\Omega^p(\mathrm{ad}\, E)$ by

$$(\mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2) := \frac{1}{2} \int_M \langle \mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2 \rangle * (1).$$

Definition 2.7.1. Let M be a compact, oriented Riemannian manifold, E a vector bundle over M, d_A a metric connection on E with corresponding curvature $F_A \in \Omega^2(\mathrm{ad}\, E)$. The *Yang–Mills functional* is then defined as

$$\mathcal{YM}(A) := (F_A, F_A) = \frac{1}{2} \int_M \langle F_A, F_A \rangle * (1),$$

and critical points of this functional are called *Yang–Mills connections*.

Note that under this definition, and since the $\mathcal{G}$ acts on F_A by conjugation and we are assuming that M is oriented and Riemannian so that $G \subseteq SO(n)$, we have that the Yang–Mills functional is *gauge invariant* -see, for example Theorem 4.2.1 of [27].

As mentioned in the introduction, the Yang–Mills functional is critical in dimension four in two senses. Analogously to the case of harmonic maps, where the Dirichlet energy is invariant under conformal transformations of the base manifold in dimension $n = 2$, the Yang–Mills energy is invariant under conformal transformations of M in dimension $n = 4$ - see, for example, [42]. The other sense in which the functional is critical in dimension four is that the Sobolev embedding $H^1 \hookrightarrow L^4$ is critical, although the significance of this will be discussed later in Section 3.1.

Since the space of metric connections on E is an affine space, $d_A + tB$ for $B \in \Omega^1(\mathrm{ad}\, E)$ is also a metric connection. Note that when acted on

$\sigma \in \Gamma(E)$, we have

$$\begin{aligned} F_{d_A+tB}(\sigma) &= (d_A + tB)(d_A\sigma + tB\sigma) \\ &= F_A(\sigma) + td_A(B\sigma) + tB \wedge d_A\sigma + t^2(B \wedge B)\sigma \\ &= F_A(\sigma) + t(d_AB)\sigma + t^2(B \wedge B)\sigma. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{YM}(d_A + tB)|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_M \langle F_{d_A+tB}, F_{d_A+tB}\rangle * (1) \\ &= \int_M \langle d_AB, F_A\rangle * (1) \\ &= \int_M \langle B, d_A^* F_A\rangle * (1). \end{aligned}$$

Therefore, since B was arbitrary, we have that d_A is a critical point of the Yang–Mills functional if and only if

$$d_A^* F_A = 0. \tag{2.9}$$

Note that this is a second order nonlinear PDE, and, when written out in co-ordinates it reads

$$\frac{\partial F_{ij}}{\partial x^i} + [A_i, F_{ij}] = 0.$$

Note that by the gauge invariance of solutions the Yang–Mills equation is non-elliptic.

2.8 Electromagnetism as an Abelian Gauge Theory

As we have previously mentioned, Yang–Mills theory arose as an attempt to generalise classical electromagnetism to more abstract settings. As such, it is instructive to recast electromagnetism as an abelian gauge theory, both to make contact with classical literature, and also to get a concrete idea of what a gauge theory looks like in the wild.

We will consider the principal bundle approach, since this is a physical example and considering the vector bundle approach would add additional complications which are unnecessary in demonstrating the theory. We refer the reader who is interested in learning more about the physical significance of vector bundles to Chapter 2.2 - 2.4 of [35].

Assuming that gravitational effects are neglected, the setting for classical electromagnetism is Minkowski spacetime, $\mathbb{R}^{1,3}$. As a differentiable manifold, this is diffeomorphic to $\mathbb{R}^4$, although it does not have a Riemannian manifold structure, but rather a pseudo-Riemannian one. Relative to the standard basis of $\mathbb{R}^{1,3}$, $\{x^0, x^1, x^2, x^3\}$, we define the pseudo-Riemannian metric $\eta_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where

$$\eta = (\eta_{\alpha\beta}) = \eta^{-1} = (\eta^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Consider then the principal fibre bundle $U(1) \hookrightarrow P \to M$, where M is a submanifold of $\mathbb{R}^{1,3}$ with the metric induced on M by restriction. If we identify $U(1)$ with the unit circle in $\mathbb{C}$, then its Lie algebra is $\mathfrak{u}(1) \simeq \operatorname{Im}\mathbb{C}$. Now, suppose that M has an open cover, and let $U_\alpha \cup U_\beta \neq \emptyset$ and

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$$

be the corresponding transition function. Recall from the transformation law for local connection forms, equation (2.8), that

$$\omega_\beta = \psi_{\beta\alpha} \omega_\alpha \psi_{\alpha\beta} + \psi_{\beta\alpha} d\psi_{\alpha\beta}.$$

Since an element of $U(1)$ can be written as $\psi_{\alpha\beta}(x) = e^{-i\Lambda_{\alpha\beta}}$, where $\Lambda_{\alpha\beta} : M \to \mathfrak{u}(1) \simeq \operatorname{Im}\mathbb{C}$ valued function. Since $U(1)$ is abelian, we have that

$$\omega_\beta = \omega_\alpha - i d\Lambda_{\alpha\beta}.$$

We may think of $\Lambda = if_{\alpha\beta}$, where $f_{\alpha\beta} : M \to \mathbb{R}$ is a real-valued function, and rewrite the transformation law as

$$\omega_\beta = \omega_\alpha + df_{\alpha\beta}, \tag{2.10}$$

which should be familiar as the usual transformation laws for potentials in Maxwell's equations. Although in Maxwell's equations we have that there is a scalar and a vector potential, in gauge theory we have accomodated for this by considering a four-manifold with the Minkowski metric.

Since $\mathfrak{u}(1)$ is abelian, we have that $\omega_\alpha \wedge \omega_\alpha = 0$, and that the corresponding local curvature form is given by

$$\Omega_\alpha = d\omega_\alpha.$$

As a peculiarity of electromagnetism by virtue of the abelian gauge group, we have that the local curvature forms agree on overlapping regions, not just transform accordingly. This means that we have

$$\Omega_\alpha = \Omega_\beta$$

on $U_\alpha \cap U_\beta$. This can be seen from the transformation law for local curvature forms, equation (2.6), or one can simply look at the transformation law for local curvature forms of electromagnetism, equation (2.10), and recall that

$d^2 f_{\alpha\beta} = 0$ for any $f_{\alpha\beta}$, and so we have that

$$\Omega_\alpha = d\omega_\alpha = d\omega_\beta = \Omega_\beta.$$

If it is clear which co-ordinate patch we are working in, we will write $\mathcal{F} := \Omega_\alpha$ and $\mathcal{A} := \omega_\alpha$, to be able to drop the reference to the co-ordinate patch whilst avoiding ambiguity as to whether we mean the local or global connection and curvature forms. Writing out the local curvature form in local co-ordinates, we have

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij} dx^i \wedge dx^j,$$

where

$$\mathcal{F}_{ij} = \frac{\partial \mathcal{A}_j}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^j}. \tag{2.11}$$

This should be familiar as the *Faraday tensor* of electromagnetism. This is a skew-symmetric two-form, and to make further contact with the physics literature, we will write $\vec{E} = (E^1, E^2, E^3)$, $\vec{B} = (B^1, B^2, B^3)$, and

$$\begin{aligned} \mathcal{F}_{i0} &= E_i \\ \mathcal{F}_{ij} &= \varepsilon_{ijk} B^k, \end{aligned}$$

so that we find

$$(\mathcal{F}_{ij}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{2.12}$$

and the dual is given by $\mathcal{F}^{ij} = \eta^{ik}\eta^{j\ell}\mathcal{F}_{k\ell}$, so that

$$(\mathcal{F}^{ij}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

One can verify that

$$**\mathcal{F} = -\mathcal{F},$$

so that the curvature form is *anti self-dual*. Such a property is particularly important in the topological aspects of Yang–Mills theory, although any further discussion would lead us far astray, and so we refer the interested reader to, for example, Chapter 1.1.5 of [15] for a readable discussion on self-dual and anti self-dual curvature forms and its significance.

Furthermore, one can easily check that

$$\begin{aligned} \frac{1}{2}\mathcal{F}_{ij}\mathcal{F}^{ij} &= |\vec{B}|^2 - |\vec{E}|^2 \\ \frac{1}{4}\mathcal{F}_{ij} * \mathcal{F}^{ij} &= \vec{E}\cdot\vec{B} \end{aligned}$$

are the two invariants of classical electromagnetic theory. Since $U(1)$ is abelian, the exterior covariant derivative acts as d, the Yang–Mills equations are

$$\begin{aligned} d^*\mathcal{F} &= 0 \\ d\mathcal{F} &= 0, \end{aligned}$$

which in local co-ordinates is written as

$$\begin{aligned} \frac{\partial \mathcal{F}^{ij}}{\partial x^i} &= 0 \\ \frac{\partial (*\mathcal{F}^{ij})}{\partial x^i} &= 0. \end{aligned}$$

It is then purely computational to check that this yields

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial x^0} &= \vec{0} \\ \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial x^0} &= \vec{0} \\ \vec{\nabla} \cdot \vec{B} &= 0, \end{aligned}$$

which are, of course, the source-free Maxwell's equations in a vacuum.

2.8.1 A Static Electric Charge

Now that we have developed electromagnetism as a gauge theory, let us briefly review the case of an electric point charge. In classical electromagnetism, we have that a scalar potential, φ, and a vector potential $\vec{A}$, which we write as $(\varphi, \vec{A})$. Recall that the potentials are given by

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial x^0} - \nabla\varphi \\ \vec{B} &= \nabla \times \vec{A}. \end{aligned}$$

For a static point electric charge, we have that $\vec{B} = 0$ and $\vec{E} = -\nabla\varphi$. In the language of gauge theory, we say that

$$\mathcal{A} = \mathcal{A}_i dx^i = \frac{-n}{r} dx^0,$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $n \in \mathbb{N}$. The Faraday tensor is then given by

$$\mathcal{F} = \frac{n}{r^2}(x^1 dx^1 + x^2 dx^2 + x^3 dx^3) \wedge dx^0,$$

and so

$$\vec{B} = 0 \qquad \vec{E} = \frac{n}{r^2}(x^1, x^2, x^3).$$

Now, let $M := \mathbb{R}^{1,3} - \{(x^0, 0, 0, 0) \in \mathbb{R}^{1,3} : x^0 \in \mathbb{R}\}$ and consider P, a principal $U(1)$ bundle over M. We are able to stay in the realm of gauge theory if we introduce *characteristic classes*, although such a detour would be significant and unnecessary, and so we will stay with the notation of classical electromagnetism and vector calculus. Now, it is clear that if $\vec{E} = -\nabla\varphi$ for some scalar potential φ, then

$$\nabla \times \vec{E} = 0$$

automatically, so finding a scalar potential for $\vec{E}$ boils down to solving Laplace's equation

$$\Delta\varphi = 0$$

on $\mathbb{R}^{1,3}$, and we know that this has solutions. We therefore see that having $\vec{\nabla}\cdot\vec{E} = 0$ is a necessary *and* sufficient condition for the existence of a globally defined scalar potential, which means that $\mathcal{A}$ is globally defined, and so there exists a globally defined section of P such that $\mathcal{A} = s^*\omega$, where ω is a connection on P. This means that the principal fibre bundle of a point electric charge *must be trivial*. In this way, the study of electric charges are not topological, since all fields are represented by connections on the trivial bundle.

2.8.2 Dirac's Magnetic Monopole

Although a point electric charge is an excellent example, it does not reflect the fundamentally topological nature of gauge theory, since the principal bundle is necessarily trivial. This is not the case in general, and we find that Dirac's magnetic monopole is an instructive example of how topology plays a crucial role in gauge theory. It will aso bring together some of the theory which has been developed as motivating examples - in particular the $S^1 \hookrightarrow S^3 \to S^2$ Hopf fibration.

In 1931, Dirac considered the case of a point magnetic charge analogous to the point electric charge of an electron in his paper [12] and further developed his ideas in [13]. In view of Coulomb's law of for static electric point charges, he noticed that a point *magnetic* charge at $(0, 0, 0) \in \mathbb{R}^3$ defined by

$$\vec{E} = 0, \qquad \vec{B} = \frac{g}{r^2}\hat{e}_r,\ r \neq 0, \tag{2.13}$$

where (r, θ, ϕ) are the standard spherical co-ordinates in $\mathbb{R}^3 - \{0, 0, 0\}$, and g is a constant, the strength of the magnetic monopole. Now, $\vec{B}$ is determined by a *vector potential* such that

$$\vec{B} = \nabla \times \vec{A}.$$

Unlike the case for $\vec{E}$, such a condition is not guaranteed. $\vec{\nabla}\cdot\vec{B} = 0$ is clearly necessary, since $\vec{\nabla} \cdot (\vec{\nabla} \times A) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = 0$ automatically. On the other hand, it is certainly not sufficient, even on a simply connected domain, like $\mathbb{R}^{1,3}$. One can see this by assuming that it is necessary and then deriving a contradiction by comparing the flux across the sphere S^2 calculated directly, and calculated using Stokes' theorem. For this calculation, see, for example, Chapter 0.2 of [36].

It is possible, however, to *locally* define vector potentials. Recall the curvature matrix, equation (2.12), and set $E^i = 0$, we then have

$$(\mathcal{F}_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x^3 & -x^2 \\ 0 & -x^3 & 0 & x^1 \\ 0 & x^2 & -x^1 & 0 \end{pmatrix},$$

recalling that we had only substituted $\vec{B} = (B^1, B^2, B^3)$ to make further contact with the physics literature. The Faraday tensor is then

$$\mathcal{F} = \frac{g}{r^2}(x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 + x^3 dx^1 \wedge dx^3).$$

Since $\mathcal{F}$ is independent of x^0, we have that the equations (2.13) define a 2-form of this type on any x^0 =constant slice of M. Moreover, if we write (x^1, x^2, x^3) in polar co-ordinates (ρ, θ, ϕ) we may write

$$\mathcal{F} = g \sin\phi d\phi \wedge d\theta.$$

Since $\mathcal{F}$ is independent of r, we may assume without loss of generality that $r = 1$, and so we may restrict our domain to S^2, in, for example, the slice $x^0 = 0$. We may then define potentials $\mathcal{A}_N$ and $\mathcal{A}_S$ on $U_N := S^2 - \{0, 0, -1\}$ and $U_S := S^2 - \{0, 0, 1\}$, respectively. On these charts, we have that

$$\begin{aligned} \mathcal{A}_N &= g(1 - \cos\phi)d\theta \\ \mathcal{A}_S &= -g(1 + \cos\phi)d\theta, \end{aligned}$$

and it is easy to check that $d\mathcal{A}_N = d\mathcal{A}_S = \mathcal{F}$. Now, to describe the transformation behaviour, note firstly that

$$\mathcal{A}_N - \mathcal{A}_S = 2gd\theta,$$

and we can use this to determine the transition function $\psi_{NS} : U_N \cap U_S \to U(1)$ by comparing it to the transformation behaviour

$$A_N = A_S + \psi_{SN} d\psi_{NS}.$$

We therefore have that

$$\psi_{NS} = e^{-2g\theta i}.$$

Since we are on the sphere, we must have 2π periodicity in θ. Namely, we require that

$$e^{-2g\theta i} = e^{-2g(\theta + 2\pi)i} \implies g = \frac{n}{2}$$

for any $n \in \mathbb{Z}$. This is *Dirac's quantisation condition*. Each value of n will produce a different transition function, and thus a different principal fibre bundle. Because of this, the value of n is call the *topological charge*. For the case $n = 1$ we have the transition function $\psi NS = e^{-\theta i}$, and this exactly induces the Hopf bundle, although we refer to, for example, Chapter 2.2 of [35] for the explicit construction.

2.9 A Word on Notation

The underlying theme in this thesis is *finding good gauges*, although the context in which we do this varies greatly. In the first two chapters we aim to find the best representative of a gauge equivalence class, and we denote a connection by $d_A := d + A$ since we are able to work locally and then patch these results together to form a global result. In the global setting, writing $D := d_A = d + A$ is an abuse of notation, and we mean that in any co-ordinate neighbourhood U_α we have that $D = d + A_\alpha$, and if $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a transformation $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$A_\beta = \psi_{\beta\alpha} A_\alpha \psi_{\alpha\beta} + \psi_{\beta\alpha} d\psi_{\alpha\beta}$$

on $U_\alpha \cap U_\beta$.

This is a quite different setting to Chapter 5 , where we aim to minimise the Yang–Mills energy by deforming an initial connection along the lines of steepest descent of the functional. In this case it is not possible to work locally and then patch the result together. This is because if it could be possible to deform A_α and A_β on U_α, U_β respectively, but it may not be possible to find a gauge change on $U_\alpha \cap U_\beta$, and so it is not possible to patch these connections together. Due to this, will will adopt the abuse of notation and write d_A when we either work locally, or we are able to work locally and then patch these local results together. Otherwise, we will denote a connection D to emphasise that it is a necessarily globally defined object.

2.10 Sobolev Spaces of Maps

Throughout this thesis we will assume the minimal regularity conditions on the connection forms and gauge changes. As such, we will need to introduce the idea of the *Sobolev space of maps*. For a review of classical Sobolev spaces, we refer the reader to, for example, Chapter 5 of [17], or Chapter 7 of [52]. When generalising the notion of a Sobolev space to maps, it is no longer clear what is meant by a derivative, and so we must fix a covariant derivative with respect to which we will consider the Sobolev space of maps, which we call ∇_{ref}. Firstly, we define the $W^{k,p}$ norm for forms taking their value in the endomorphism bundle to be:

$$||A||_{W^{k,p}(\Omega^i(\text{ad } E))} := \left(\sum_{\ell=0}^{k} ||\nabla^\ell_{\text{ref}} A||^p_{L^p} \right)^{\frac{1}{p}} < \infty,$$

where A is a i-form with measurable co-efficients which takes its values in ad E. Note that we do not say directly that $A \in \Omega^i(\text{ad } E)$, since elements of $\Omega^i(\text{ad } E)$ are defined to be smooth. This definition is independent of of the choice of ∇_{ref}. With this notation, we may then say that $D \in W^{k,p}$ if

$D = D_{\text{ref}} + A$, where $D_{\text{ref}} \in \Omega^1(\text{ad } E)$ and $A \in W^{k,p}(\Omega^1(\text{ad } E))$. The appropriate notion of the Sobolev space of gauge transforms is slightly more complicated. We must consider gauge transforms as maps between manifolds, which means that we must identify $G \subseteq SO(n) \subseteq \mathbb{R}^{n\times n}$ via a representation. Furthermore, since we may only identify a gauge transform as $S : U \to G$ locally, we are only able to define the Sobolev space of gauge transforms locally. Formally, to define the Sobolev space of gauge transforms, consider a cover $\{U_\alpha\}$ of M. We then define

$$\mathcal{G}^{k,p}(M) := \{S : \ S|_{U_\alpha} \in W^{k,p}(U_\alpha, \mathbb{R}^{n\times n}) \text{ and } S(x) \in G \text{ a.e.}\}. \tag{2.14}$$

For an in-depth discussion on Sobolev spaces of maps between manifolds we refer the reader to [5]. For a further discussion on the Sobolev gauge group specifically, we refer to Chapter 20.1 of [18] or Appendix A of [19]. Since the space of connections is an affine space, we define the Sobolev space of connections to be the affine space

$$\mathcal{A}^{k,p}(M) = \{d_A = d + A : \ A \in W^{k,p}(\Omega^1(\text{ad } E))\}. \tag{2.15}$$

Since we will frequently be considering the product of two (or more) elements of Sobolev spaces, the Sobolev multiplication theorems will be of fundamental importance to our analysis.

Theorem 2.10.1 (Sobolev Multiplication Theorem)**.** *Let k_i, k and $1 \leq p \leq p_i < \infty$ for $i = 1, 2$ be real numbers satisfying*

(i) $k_1 \geq k \geq 0$,

(ii) $k \in \mathbb{N}_0$,

(iii) $k_i - k \geq n$,

(iv) $k_1 + k_2 - k > \frac{1}{n}(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0$.

The strictness of the inequalities in (iii) and (iv) can be interchanged. Then, the pointwise multiplication of functions is a continuous bilinear map

$$W^{k_1,p_1} \times W^{k_2,p_2} \to W^{k,p}.$$

Amongst other places, a proof of this theorem can be found at [2]. For $p(k+1) > n$, the space $\mathcal{G}^{k+1,p}$ is a smooth (infinite dimensional) manifold and a (Banach) Lie group. Unfortunately, the proof of this is beyond our means, and so we refer to Lemma 4.4.3 of [20] and Proposition A.2 of [19] for a proof. In the same Sobolev range, we have by the Sobolev embedding theorems that $W^{k+1,p}(M, \text{Aut } E) \hookrightarrow C^0(M, \text{Aut } E)$, see, for example, Theorem B.2 of [57]. This has the important consequence that multiplication and inversion are continuous maps on $\mathcal{G}^{k+1,p}$ - see, for instance, Lemma A.5 of [57]. Such a condition guarantees the preservation of topology under gauge transformations. By this, we mean that we will not induce singularities when changing gauge- something which is not guaranteed when we have $p(k+1) \leq n$. This problem (and its resolution) will in fact occupy us for the entirety of Chapter 4 for the case when $n = 4$, $p = 2$ and $k = 1$. Furthermore, for the range $p(k+1) > n$, the group of gauge transformations acts smoothly on the space of connections. We prove this in the case where $k = 1$, as this is the case of most interest to us.

Lemma 2.10.2. *For $2p > n$, the induced map*

$$\mathcal{G}^{2,p} \times \mathcal{A}^{1,p} \to \mathcal{A}^{1,p}$$
$$(S, d + A) \mapsto d + S^{-1}ds + S^{-1}As$$

is smooth. Furthermore, if $d_A = S^{-1} \circ d_{\tilde{A}} \circ S$ for $d_A, d_{\tilde{A}} \in \mathcal{A}^{1,p}$, then $S \in \mathcal{G}^{2,p}$.

Proof. From the multiplication theorem for Sobolev spaces, Theorem 2.10.1, we see that the map

$$W^{2,p}(M, \mathrm{Aut}\, E) \times W^{1,p}(M, T^*M \otimes \mathfrak{g}) \to W^{1,p}(M, T^*M \otimes \mathfrak{g})$$

is continuous for $2p > n$, and so the map $A \mapsto S^{-1}dS + S^{-1}AS$ is continuous. From the continuity of this map, it follows from the Banach manifold structure of $\mathcal{G}^{2,p}$ that the map is then smooth. Unfortunately, the proof of this is beyond our means, and so we refer to Theorem 4.4.3 of [20] for a proof.

Now, suppose that $d_A = S^{-1} \circ d_{\tilde{A}} \circ S$, for $d_A, d_{\tilde{A}} \in \mathcal{A}^{1,p}$, then $A = S^{-1}dS + S^{-1}\tilde{A}S$, or rather, $dS = SA - \tilde{A}S$. By the Sobolev multiplication theorems, we know that $SA, \tilde{A}S \in W^{1,p}$, and so we have that $dS \in W^{1,p}$, with the estimate $||dS||_{W^{1,p}} \leq ||SA||_{W^{1,p}} + ||\tilde{A}S||_{W^{1,p}}$. Therefore $S \in \mathcal{G}^{2,p}$. □

Chapter 3

Gauge Construction

3.1 Motivation

Before moving on to the main business of this chapter, we pause briefly to motivate the problem and review the significance of finding a solution. As always, let M be a compact, finite dimensional, boundary-free, oriented Riemannian manifold and E a vector bundle of rank ℓ over M. Consider the functional

$$\mathcal{YM}_p(A) = \int_M |F_A|^p * (1) = \int_M |dA + A \wedge A|^p * (1) = ||F_A||^p_{L^p}, \qquad (3.1)$$

which we would like to minimise. The L^p norm is sub-additive, so we have that

$$||F_A||^p_{L^p} \leq C\big(||dA||^p_{L^p} + ||A \wedge A||^p_{L^p}\big) \leq C\big(||dA||^p_{L^p} + ||A||^{2p}_{L^{2p}}\big),$$

which implies that we require $A \in W^{1,p} \cap L^{2p}$ a priori. To apply classical calculus of variations theory we require that the functional be coercive, which means that $\mathcal{YM}_p(A)$ controls both the $W^{1,p}$ and the L^{2p} norms. For the range $2p \geq n$, however, this situation simplifies significantly, thanks to the Sobolev embedding theorem.

Lemma 3.1.1. *For an n-dimensional base manifold and $2p \geq n$, we have the following continuous embedding:*

$$W^{1,p} \hookrightarrow L^{2p},$$

with the estimate

$$||A||_{L^{2p}} \leq C||A||_{W^{1,p}}.$$

Moreover, for $2p > n$, this embedding is compact.

Proof. By the Sobolev embedding theorem, we have that

$$W^{1,p} \hookrightarrow L^{p^*},$$

with the estimate

$$||A||_{L^{p^*}} \leq C||A||_{W^{1,p}},$$

where if we demand that $p^* = 2p$, we have

$$\frac{1}{p^*} \geq \frac{1}{p} - \frac{1}{n} \implies 2p \geq n.$$

For $2p > n$, this embedding is compact by the Rellich-Kondrachov compactness theorem. $\square$

The above embedding is critical precisely when $2p = n$, and although we will discuss the Yang–Mills functional with exponent p in this chapter, we note that in the case $p = 2$, this makes the embedding critical in dimension four. Due to Lemma 3.1.1, the coercivity condition in the range $2p \geq n$ simplifies to be

$$||A||_{W^{1,p}} \leq C(E)\mathcal{YM}_p(A),$$

since if we have control the the $W^{1,p}$ norm, then we automatically have control the the L^{2p} norm. As another consequence of the Sobolev embedding theorem, we see that for $2p > n$, we have the compact inclusion

$$W^{2,p} \hookrightarrow C^0,$$

although we refer to, for example, [57], Lemma $B.3$ (iii) for the proof. This embedding fails for $2p = n$, although we still have the embedding $W^{2,p} \hookrightarrow VMO$, the space of vanishing mean oscillations (see [11], example 2 of chapter I.2), although we do not exploit this in this chapter.

Unfortunately, the functional $\mathcal{YM}_p(A)$ is degenerate since its second variation is a non-elliptic operator (see, for example, [7] or [6] for an explicit calculation). The source of this non-ellipticity is the infinite dimensional group of gauge transformations under which the functional is invariant, its *symmetry group*. A discussion on this can be found in [9], chapter I, Section (iv), but the main idea is that since any solution is gauge-equivalent to infinitely many other solutions, the operator $\mathcal{YM}_p(\cdot)$ has an infinite dimensional kernel, and so cannot be an elliptic operator. Since a variation along a gauge orbit will produce another solution to the equation, the class of admissable variations must be reduced if there is a hope of applying classical variational methods to the problem. In view of the Hodge decomposition, see, for example, Theorem 5.2 of [56], and the discussion in [8], it turns out that variations of the functional whose tangent vectors (as an element of $\Omega^1(\text{ad } E)$) lie in the image of the mapping $d^* : \Omega^1(\text{ad } E) \to \Omega^0(\text{ad } E)$ are *transverse* to gauge orbits. In light of this, we replace $\mathcal{YM}_p(A)$ with the functional

$$E_p(A) = \int_M \left(|F_A|^p + |d^*A|^p\right) * (1) \geq \mathcal{YM}_p(A).$$

This approach is common in literature - see, for example, [41], chapter 1.2. This functional is non degenerate since its second variation is positive definite, see, for example [6]. Heuristically, we can see that this is elliptic becuase gauge equivalent variationts are no longer admissable since we have *broken the gauge symmetry* and thus reduced the gauge group. We then aim to minimise $E_p(A)$, whilst demanding equality to ensure that a minimiser of $E_p(A)$ is also a minimiser of $\mathcal{YM}_p(A)$. This approach is common in geometric analysis, and was first employed to solve Plateau's problem. For more on this, see Riviere's exposition at the start of [44]. It is clear that these two functionals are equal if and only if $d^*A = 0$, which is called the *Coulomb gauge condition*. By enforcing the Coulomb gauge condition, we are fixing a gauge, which means to choose a representative from each gauge orbit. The Coulomb gauge will be a recurring object in this thesis, and finding good gauges will often be equivalent to finding a Coulomb gauge.

It is clear then, that the major questions to be answered are:

(i) Does every connection form possess a Coulomb gauge representative?

(ii) When does $\mathcal{YM}_p(A)$ control $||A||_{W^{1,p}}$?

It is Uhlenbeck's 1982 paper 'Connections with L^p Bounds on Curvature', [53], which gives a definitive answer to both questions for the range $2p \geq n$. It is our intention to give an exposition of her now classical method for local gauge construction in Section 3.2, and in Section 3.3 we piece these local theorems together to yield a global result.

3.2 Uhlenbeck's Local Gauge Construction Method

Here we present the result of Uhlenbeck which stipulates conditions on the $L^{\frac{n}{2}}$ norm of the curvature to gaurantee the existence of a connection which satisfies the Coulomb condition, and whose $W^{1,p}$ norm is controlled by the L^p norm of the curvature in the range $2p \geq n$. Since we are working locally, we may assume that $M = B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the base manifold. Of the space of Sobolev connections as defined in 2.15 we have ths distinguished subspace

$$\mathcal{A}_\kappa^{k,p}(B^n) = \left\{ D \in \mathcal{A}^{k,p} : \int_{|x|\leq 1} |F_A|^{\frac{n}{2}} * (1) \leq \kappa \right\}.$$

We also require that the gauge group has one more degree of differentiability than the connections, i.e., we are working with $\mathcal{G}^{k+1,p}(B^n)$. For $p(k+1) > n$, the group $\mathcal{G}^{k+1,p}$ is a smooth (infinite dimensional) manifold and a (Banach) Lie group. Unfortunately, the proof of this is beyond our means, and so we refer to Lemma 4.4.3 of [20]. As previously mentioned, in this range we have the embedding $W^{k+1,p}(M, \text{Aut }\eta) \hookrightarrow C^0(M, \text{Aut }\eta)$, which has the important consequence that multiplication and inversion are continuous maps on $\mathcal{G}^{k+1,p}$ - see, for instance, Lemma A.5 of [57]. Such a condition guarantees the preservation of topology under gauge transformations. We now state the main theorem of this section.

Theorem 3.2.1. *Let $n > p > \frac{n}{2}$ and assume $G \subseteq SO(\ell)$ is compact. Then there exists $\kappa = \kappa(E) > 0$ and $C = C(E)$ such that for every connection $D \in \mathcal{A}_\kappa^{1,p}$ there exists an $S \in \mathcal{G}^{2,p}$ such that $S^{-1} \circ D \circ S = d + A$ and:*

(i) $d^* A = 0$

(ii) $*A = 0$ *on* $S^{n-1} = \partial B^n$

(iii) $||A||_{W^{1,\frac{n}{2}}} \leq C(E) \left(\int_{|x|\leq 1} |F_A|^{n/2} * (1) \right)^{2/n}$

(iv) $||A||_{W^{1,p}} \leq C(E) \left(\int_{|x|\leq 1} |F_A|^{p} * (1) \right)^{1/p}$.

Remark 3.2.2. It is crucial that $2p > n$ for this lemma, otherwise the the elements of $\mathcal{G}^{2,p}(B^n)$ are not necessarily continuous and $\mathcal{G}^{2,p}(B^n)$ does not act smoothly on $\mathcal{A}^{1,p}(B^n)$. It is our intention, however, to extend this theorem to include the case $p = \frac{n}{2}$ by using an approximating sequence and limit argument.

Before we give the proof, we show that inequalities (iii),(iv) of Theorem 3.2.1 are a priori estimates of solutions which satisfy (i),(ii).

Lemma 3.2.3. *There exists $k(E) > 0$ such that if $||A||_{L^n} \le k(E)$ and (i),(ii) of Theorem 3.2.1 are satisfied for $n > p \ge \frac{n}{2}$, then*

$$||A||_{W^{1,p}} \le C(E)\left(\int_{|x|\le 1} \left|F_A\right|^p * (1)\right)^{\frac{1}{p}}.$$

Proof. The system of equations

$$\begin{cases} F_A = dA + A \wedge A \text{ in } B^n \\ d^*A = 0 \text{ in } B^n \\ *A = 0 \text{ on } S^{n-1} \end{cases}$$

is an overdetermined elliptic system with Neumann boundary conditions. We may use elliptic estimates from Hodge theory, see, for example, Theorem 5.1 of [57] This yields

$$||A||_{W^{1,p}} \le k'(E)||dA||_{L^p},$$

since A is in Coulomb gauge and the domain is simply connected. From the equation $F_A = dA + A \wedge A$ we see that

$$||dA||_{L^p} \le ||F||_{L^p} + ||A||^2_{L^{2p}}$$

And by noting that $1 = p/q + p/n = \frac{1}{q/p} + \frac{1}{n/p}$ we see from the Hölder inequality that

$$||A||^2_{L^{2p}} \le ||A||_{L^n}||A||_{L^q}.$$

We also have the Sobolev inequality

$$||A||_{L^q} \le C||A||_{W^{1,p}}.$$

Putting the last four inequalities together, we get

$$(1 - k'(E)C||A||_{L^n})||A||_{W^{1,p}} \le k'(E)||F||_{L^p},$$

and the result follows for $||A||_{L^n} \le \frac{1}{2}\frac{1}{Ck'(nE)}$. □

Remark 3.2.4. This is the first point where it is important that the curvature be small. Without this smallness assumption, it wouldn't necessarily be true that estimates (iii) & (iv) follow from (i) & (ii), which is one of the main goals of Uhlenbeck's gauge construction method.

Proof of Theorem 3.2.1. The proof of this theorem proceeds in three main parts. Firstly, we will show that $\mathcal{A}^{k,p}_\kappa$ is connected, then that the set of connections in $\mathcal{A}_\kappa^{k,p}$ which satisfy $(i)-(iv)$ is both open and closed, and so is all of $\mathcal{A}_\kappa^{k,p}$, since it is non-empty.

Claim 3.2.5. *$\mathcal{A}_\kappa^{k,p} \subseteq \mathcal{A}^{k,p}$ is connected for $n > p > \frac{n}{2}$.*

Proof. Choose an arbitrary $D \in \mathcal{A}_\kappa^{k,p}$ such that $D = d + A$. We then define the one parameter family $D_\sigma = d + \sigma A(\sigma x)$. Note that since $D_0 = d$ and $F_d = 0$, we have $d \in \mathcal{A}_\kappa^{k,p}$. The curvature of $F_{D_\sigma}(x)$ is then given by

$$F_{D_\sigma}(x) = \sigma^2(dA)(\sigma x) + \sigma^2[A(\sigma x), A(\sigma x)] = \sigma^2 F_D(\sigma x)$$
$$||F_{D_\sigma}(x)||_{n/2,0}^{n/2} = \int_{|x|\leq 1} |F_{D_\sigma}(x)|^{n/2} * (1) = \int_{|x|\leq\sigma} |F_D|^{n/2} * (1) \leq \kappa.$$

So we see for $\sigma \in [0,1]$ and $2p \geq n$ that $D_\sigma \in \mathcal{A}_\kappa^{k,p}$. We have constructed a continuous curve in $\mathcal{A}_\kappa^{k,p}$, and since $D = d + A$ was arbitrary, it follows that $\mathcal{A}_\kappa^{k,p}$ is connected. □

Claim 3.2.6. *The set $D \in \mathcal{A}_\kappa^{k,p}$ such that there exists $s \in \mathcal{G}^{2,p}$, where $S^{-1} \circ D \circ S = d + A$ and A satisfies (i)-(iv) of the Theorem is closed.*

Proof. Let $D_i = d + \tilde{A}_i \to d + \tilde{A} \in \mathcal{A}_\kappa^{k,p}$ be a sequence of connections convergent in $\mathcal{A}_\kappa^{k,p}$ such that D_i is gauge equivalent to $d + A_i$, where conditions $(i)-(iv)$ hold on A_i. Firstly, note that

$$||F_{\tilde{A}_i}||_{L^p} \leq C(||\tilde{A}_i||_{W^{1,p}} + ||\tilde{A}_i||_{L^{2p}}^2) \leq C',$$

where C' is a uniform constant, since $2p > n$. This follows since $\tilde{A}_i$ converge to $\tilde{A}$ in $W^{1,p}$. Then, since $\tilde{A}_i$ and A_i are gauge equivalent and each A_i satisfies (i)-(iv), we have

$$||A_i||_{W^{1,p}} \leq C(E)||F_{A_i}||_{L^p} \leq C(E)C'.$$

Therefore the sequence A_i is bounded uniformly in $W^{1,p}$, and so by the Banach-Alaoglu theorem, there exists a weakly convergent subsequence (also indexed by i) such that $A_i \rightharpoonup A \in W^{1,p}(B^n, T^*M \otimes \mathfrak{g})$. Consider

$$dS_i = S_i A_i - \tilde{A}_i S_i.$$

For $1/n - 1/p + 1/q = 0$, since s_i is orthogonal we get that

$$||dS_i||_{L^q} \leq ||A_i||_{L^q} + ||\tilde{A}||_{L^q} \leq C(||A_i||_{W^{1,p}} + ||\tilde{A}||_{W^{1,p}})$$

by the Sobolev inequality. Since G is compact, $||S_i||_{W^{1,q}(B^n,\mathfrak{g})}$ is uniformly bounded, and so there exists a weakly convergent subsequence (also indexed by i) $S_i \rightharpoonup S$ in $W^{1,q}(B^n, \mathfrak{g})$. We know that weak limits are preserved by linear equations, and so we see that

$$dS_i = S_i A_i - \tilde{A}_i S_i \to SA - \tilde{A}S = dS.$$

From Lemma 2.10.2, we have that $S \in \mathcal{G}^{2,p}$, and so $d + \tilde{A}$ is gauge equivalent to $d + A \in \mathcal{A}^{k,p}$, and also by the preservation of linear equations under weak limits we have that this satisfies (i)-(iv) of the theorem. Therefore the space in $\mathcal{A}_\kappa^{k,p}$ such that $d+\tilde{A}$ is gauge equivalent to $d+A$, where A satisfies (i)-(iv) is closed. $\square$

We require the following lemma in preparation for the openness result. This lemma will allow us to assume that the boundary conditions when we apply the implicit function theorem are homoegenous, so as to avoid the need to use boundary value spaces.

Claim 3.2.7. *There exists a linear operator $P : W^{1,p}(B^n) \to W^{2,p}(B^n)$ such that if $f \in W^{1,p}(B^n)$, $P(f) \in W^{2,p}(B^n)$, $P(f)|_{S^{n-1}} = 0$ and $(*dP(f) + f)|_{S^{n-1}} = 0$.*

Proof. Let $P(f)$ be the solution of a heat equation $0 < r \leq 1$ with zero initial conditions at $r = 1$ and inhomogeneous Neumann boundary conditions, multiplied by a smooth cut-off function ϕ with $\phi(0) = 0$ and $\phi(x) = 1$ near $|x| = 1$. Invert the heat operator with r as time, S^{n-1} as space. Then

$$\begin{aligned} P(f) &= \phi[\frac{\partial}{\partial r} - \Delta_{S^{n-1}}]^{-1} f \\ P(f)|_{r=1} &= 0 \\ *dP(f) &= -f \qquad \text{on } S^{n-1} \end{aligned}$$

Standard L^p regularity theory for the heat equation gives $P(f) \in W^{2,p}(B^n - \{0\})$ for $f \in W^{1,p}(B^n - \{0\})$. $\square$

We are now in a position where it is possible to use an implicit function theorem argument. Suppose that we have a connection $D \in \mathcal{A}_\kappa^{1,p}$ which is gauge equivalent to $d + A \in \mathcal{A}^{1,p}$, where A satisfies (i)-(iv). Then we want to show that there is an open set around D in $\mathcal{A}_\kappa^{1,p}$ where each element in this open set is also gauge equivalent to an element in $\mathcal{A}^{1,p}$ which satisfies (i)-(iv). Namely, we want to show that for all $d + \tilde{A} \in \mathcal{A}_\kappa^{1,p}$, the equation

$$d^*(S^{-1}dS + S^{-1}\tilde{A}S) = d^*A = 0$$

is solved, where $*dS|_{S^{n-1}} = 0$ and $*\lambda|_{S^{n-1}} = 0$ so that $*A|_{S^{n-1}} = 0$. This leads us to the following definitions:

$$\begin{aligned} W_\nu^{1,p} &= \{\lambda \in W^{1,p}(B^n, T^*M \otimes \mathfrak{g}) : *\lambda|_{S^{n-1}} = 0\} \\ \mathcal{G}_\nu^{2,p} &= \{S \in W^{2,p}(B^n, G) : *dS|_{S^{n-1}} = 0\}. \end{aligned}$$

Claim 3.2.8. *For $n > p > \frac{n}{2}$, suppose $d + A \in \mathcal{A}^{1,p}$ such that $||A||_{L^n} \leq \kappa(E)$ satisfies (i)-(iv). Then there exists $\varepsilon > 0$ such that for $||\lambda||_{W^{1,p}} \leq \varepsilon$, $\lambda \in W_\nu^{1,p}$ the non-linear equation*

$$d^*(S^{-1}dS + S^{-1}(A + \lambda)S) = 0$$

has a solution $S(\lambda) \in \mathcal{G}_\nu^{2,p}$ and the solution depends smoothly on $\lambda \in W_\nu^{1,p}$ and also satisfies (i)-(iv).

Proof. Firstly, introduce the following spaces

$$W_\nu^{2,p\perp} = \left\{U \in W^{2,p}(B^n, \mathfrak{g}) : \int_{B_n} U * (1) = 0,\ *dU|_{S^{n-1}} = 0\right\}$$
$$L^{p\perp} = \left\{V \in L^p(B^n, \mathfrak{g}) : \int_{B^n} V * (1) = 0\right\}.$$

The stipulation $\int_{B_n} U * (1) = 0$ means that if $U \equiv$ const, then $u \equiv 0$. Then the operator

$$(U, \lambda) \mapsto d^*(e^{-U} de^U + e^{-U}(A + \lambda)e^U)$$

is a smooth map

$$W_\nu^{2,p\perp} \times W_\nu^{1,p} \to L^{p\perp},$$

where the image of the map lies in $L^{p\perp}$ by Stokes' theorem and the homogeneous boundary conditions. To linearise this operator, consider the small pertubation of U about $U, \lambda = 0$. Then

$$\begin{aligned}
&\frac{d}{dt}(e^{-U-t\psi} de^{U+t\psi} + e^{-U-t\psi}(A+\lambda)e^{U+t\psi})|_{U,\lambda=0} \\
=&(-\psi e^{-U-t\psi} de^{U+t\psi}) + e^{-U-t\psi}(d(\psi e^{U+t\psi}) - \\
&\psi e^{-U-t\psi}(A+\lambda)e^{U+t\psi} + e^{-U-t\psi}(A+\lambda)\psi e^{U+t\psi})|_{U,\lambda=0} \\
=&d\psi + [A, \psi].
\end{aligned}$$

With this, we have the self-adjoint operator

$$\begin{aligned}
H : W_\nu^{1,p} &\to L^{p\perp} \\
\psi \mapsto d^*(d\psi + [A, \psi]) &= d^* d\psi + [A, d\psi].
\end{aligned}$$

This is in fact a Banach space isomorphism, and to see this note that the operator is surjective since this is a homogeneous neumann boundary value problem, and so a solution is guaranteed to exist for each $V \in L^{p\perp}$, and the solution will have regularity $W^{2,p}$. For more background on this, see, for example, Theorem 1.5 of [57]. To see injectivity, note that in a similar method to the proof of the a-priori estimates, Lemma 3.2.3, for small $||A||_{L^n}$ and for the same p, q we have

$$\begin{aligned}
||H||_{L^p} &\geq ||d^* dA||_{L^p} - ||A||_{L^p}||d\psi||_{L^q} \\
&\geq ||d\psi||_{W^{1,p}}||(C(E) - k'(E)||A||_{L^p}) \\
&\geq 0.
\end{aligned}$$

Therefore we may then apply the implicit function theorem for Banach spaces, see, for example, Theorem E.1 of [57], and the result follows. □

Remark 3.2.9. This claim is the critical step in the proof. By the Sobolev embedding theorem, we have that for $p > \frac{n}{2}$, the map

$$W_\nu^{2,p\perp} \times W_\nu^{1,p} \to L^{p\perp}.$$

is *smooth*, since $W^{2,p} \hookrightarrow C^0$, but for $p = \frac{n}{2}$, this embedding fails, and so algebraic manipulation of an element of $W^{2,\frac{n}{2}}$ is not necessarily continuous.

Claim 3.2.10. *Suppose $D \in \mathcal{A}^{1,p}_\kappa$ for $n > p > \frac{n}{2}$ is gauge equivalent to $d + A$, where A satisfies (i)-(iv) of the theorem. Then, if κ is sufficiently small, there exists an open neighbourhood of $D \in \mathcal{A}^{1,p}_\kappa$ such that every D in this neighbourhood is gauge equivalent to a connection satisfying (i)-(iv).*

Proof. We would like to apply the result of Claim 3.2.8, but we cannot assume that $*\lambda = 0$, so we Let $U = U(\lambda) = P(*\lambda)$ where P is the linear operator constructed in Claim 3.2.7. Make the gauge transformation

$$e^{-U}(d + A + \lambda)e^U = d + e^{-U}de^U + e^{-U}Ae^U + e^{-U}\lambda e^U = d + A + \tilde{\lambda}$$

where $\tilde{\lambda} = e^{-U}Ae^U - A + e^{-U}de^U + e^{-U}\lambda e^U$ and $||U||_{W^{2,p}} \leq \tilde{c}||*\lambda||_{W^{1,p}}$ from Claim 3.2.7. It is then possible to make $||\tilde{\lambda}||_{W^{1,p}}$ arbitrarily small by making $||\lambda||_{W^{1,p}}$ sufficiently small so that $||\tilde{\lambda}|| < \varepsilon$ as in Claim 3.2.8. Since $U = 0$ on S^{n-1}, $de^U|_{U=0} = e^U dU|_{U=0} = dU$ on S^{n-1} and $*\tilde{\lambda}|_{S^{n-1}} = (*dU + *\lambda)|_{S^{n-1}} = 0$. We may now apply Claim 3.2.8 to $d + A + \tilde{\lambda}$, and this establishes an open neighbourhood about $d + A$. We may then pull back this neighbourhood by the gauge transform taking D to $d + A$ and intersect it with $\mathcal{A}^{1,p}_\kappa$, thus giving the open neighbourhood about D. □

Remark 3.2.11. Unfortunately, we are not able to apply the implicit function theorem directly to the connection D in $\mathcal{A}^{1,p}_\kappa$. This is because it was necessary that the solution $S(\lambda)$ be close to the identity so that writing $S = \exp(U + t\psi)$ makes sense and there is no guarantee that the transform from D to $d + A$ is close to the identity. This is only a slight complication, however, since the gauge group acts continuously on $\mathcal{A}^{1,p}_\kappa$ for $2p > n$. This guarantees that the pullback of an open neighbourhood in $\mathcal{A}^{1,p}$ about $d + A$ corresponds to an open neighbourhood about $D \in \mathcal{A}^{1,p}_\kappa$.

This concludes the proof, since for ε small enough we have proven that the space of connections in $\mathcal{A}^{1,p}_\kappa$ which are gauge equivalent to a connection in $\mathcal{A}^{1,p}$ which satisfies (i)-(iv) of the theorem is connected, open, closed and non-empty. Therefore, it must be the whole space, and *every* connection $D \in \mathcal{A}^{1,p}_\kappa$ is gauge equivalent to a connection in $\mathcal{A}^{1,p}$ which satisfies (i)-(iv). □

Remark 3.2.12. As noted in Claim 3.2.10, it is not necessary that the λ satisfy homogeneous Neumann boundary conditions, which is the motivation for Claim 3.2.7. A method of proof where homogeneous Neumann boundary conditions are not assumed in Claim 3.2.8 is possible, although such a method requires the use of boundary value spaces. This makes Claim 3.2.7 redundant, and Claims 3.2.8 and 3.2.10 can be condensed into one proof. This is the method adopted by [57] and [44] in their expositions of Uhlenbeck's method of local gauge construction.

As previously stated, we are able to extend the result of Theorem 3.2.1 to include the case $2p = n$ by using a limit argument.

Corollary 3.2.13. *Suppose* $d + \tilde{A} \in \mathcal{A}^{1,\frac{n}{2}}$ *and* $F_{\tilde{A}} = d\tilde{A} + \tilde{A} \wedge \tilde{A}$ *satisfies*

$$\int_{|x|\leq 1} |F_{\tilde{A}}|^{\frac{n}{2}} * (1) \leq \kappa(E).$$

Then there exists $S \in \mathcal{G}^{2,\frac{n}{2}}$ *such that* $A = S^{-1}dS + S^{-1}\tilde{A}S$ *satisfies (i)-(iv) of Theorem 3.2.1.*

Proof. As in Evans, [17], chapter 5, we may approximate $\tilde{A}$ by smooth functions such that $\tilde{A}_i \to A \in W^{2,\frac{n}{2}}(B^n, T^*M \otimes \mathfrak{g})$ with each $\int_{|x|\leq 1} |F_{\tilde{A}_i}|^{\frac{n}{2}} * (1) < \kappa(E)$. Since each $\tilde{A}_i$ is smooth, we have that Theorem 3.2.1 applies, and so each is gauge equivalent to an A_i which satisfies (i)-(iv) of Theorem 3.2.1. Since Claim 3.2.6 holds for $p = \frac{n}{2}$, we know that the space $\mathcal{A}_\kappa^{2,\frac{n}{2}}$ is closed, and so the limit of this sequence also lies in $\mathcal{A}_\kappa^{2,\frac{n}{2}}$, and the result follows. □

3.3 Weak Uhlenbeck Compactness

In the previous section we constructed only local gauges over balls whose $L^{\frac{n}{2}}$ norm of curvature is small enough, and in this section we follow Uhlenbeck's method to piece them together to give a global gauge. To do this, we suppose that M has an open cover $\{U_\alpha\}_{\alpha\in I}$ where each U_i is a ball on which the $L^{\frac{n}{2}}$ norm of curvature is sufficiently small. We then work with the local trivialisations of the gauges in each of these elements given by Theorem 3.2.1 in the hope that we will be able to construct global gauge transformations. Before we get to proving the main theorems, we give a few preliminary lemmas. In the following, φ, ϕ are overlap functions, and these represent the same bundle if there exists a subcover $V_\alpha \subseteq U_\alpha$, $M \subseteq \bigcup_\alpha V_\alpha$ and $\rho_\alpha : U_\alpha \to G$ such that $\phi_{\alpha\beta} = \rho_\alpha \varphi_{\alpha\beta} \rho_\beta^{-1}$. For the following lemma, $\exp : \mathfrak{g} \to G$ is the usual exponential map and we fix a neighbourhood $\tilde{G}$ of the identity in G in the domain of $\exp^{-1}$.

Lemma 3.3.1. *Let G be a compact group with an equivariant metric. Then there exists* $f_0 > 0$ *such that if* $f, g, \rho \in G$, $|\exp^{-1} hg| \leq f_0$ *and* $|\exp^{-1}\rho| < f_0$, *then* $\varphi\rho\phi \in \tilde{G}$ *and*

$$|\exp^{-1}\varphi\rho\phi| \leq 2(|\exp^{-1}\varphi\phi| + |\exp^{-1}\rho|)$$

Proof. The map Q given by the formula

$$\exp(Q(k,u)) = \exp k \exp u$$

is defined and smooth for (k,u) in a neighbourhood of $0 \in \mathfrak{g}$. We have $Q(0,0) = 0$ and $|dQ(0,0)| = 1$ (assuming that $k'(0) = u'(0) = 0$), since
$\frac{d}{dt}\exp_{(0,0)}(Q(k(t),u)) = k'(t)\exp k(t)\exp u|_{t=0,u=0} = 0$ and
$\frac{d}{dt}\exp_{(0,0)}(Q(k,u(t))) = u'(t)\exp u(t)\exp k|_{t=0,k=0} = 0$. Choose $\mathcal{O} = \{x \in \mathfrak{g} : |x| \leq f_0\}$ such that $|dQ(k,u)| \leq 2$ for $k,u \in \mathcal{O}$. Since $\mathcal{O}$ is convex, by the mean value theorem $|Q(k,u)| \leq 2(|k| + |u|)$ for $|k|,|u| \leq f_0$. The lemma follows if we set $k = \exp^{-1}(hg)$ and $u = g^{-1}\exp^{-1}(\rho)g = Ad_{g^{-1}}(\exp^{-1}\rho)$.

Note that

$$\begin{aligned}\exp u = \exp\bigl(g^{-1}\exp^{-1}(\rho)g\bigr) &= \sum_{k=0}^{\infty}\frac{1}{k!}(g^{-1}\exp^{-1}(\rho)g)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}g^{-1}(\exp^{-1}(\rho))^k g \\ &= g^{-1}\exp\bigl(\exp^{-1}\rho\bigr)g = g^{-1}\rho g\end{aligned}$$

With this in mind, we see that

$$\begin{aligned}Q(k,u) &= \exp^{-1}(hg\exp\bigl(Ad_{g^{-1}}(\exp^{-1}\rho)\bigr)) = \exp^{-1}(h\rho g)\\ |Q(k,u)| &\le 2(|\exp^{-1}(hg)| + |Ad_{g^{-1}}(\exp^{-1}\rho)|) = 2(|\exp^{-1}hg| + |\exp^{-1}\rho|),\end{aligned}$$

since $Ad_{g^{-1}}$ is an isometry. From this, we see the result. □

Since M is compact, for every cover there exists a finite subcover, so fix an open cover $\{U_\alpha\}$ where $\alpha \in I = \{1, 2, ..., \ell\}$. The following proposition will be used to conclude topological information about the bundle structure we are going to construct our gauges on.

Proposition 3.3.2. *Let $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ and $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be two sets of continuous transition functions describing vector bundles over M. Then there exists f_ℓ such that if*

$$m = \max_{\substack{(\alpha,\beta)\\ x\in U_\alpha\cap U_\beta}} |\exp^{-1}\left(\phi_{\alpha\beta}(x)\varphi_{\beta\alpha}(x)\right)| \le f_\ell, \tag{3.2}$$

then the following holds:

There exists a smaller cover $V_\alpha \subseteq U_\alpha$, $M \subseteq \bigcup_\alpha V_\alpha$ and continuous $\rho_\alpha : V_\alpha \to G$ such that $\phi_{\alpha\beta} = \rho_\alpha\varphi_{\alpha\beta}\rho_\beta^{-1}$ on $V_\alpha \cap V_\beta$. Morever, $\max\limits_{x\in V_\alpha} |\exp^{-1}\rho_\alpha| \le c_\ell m$.

Proof. This proof is inductive on the number of elements in the cover. For the base step, let $\ell = 1$. Then $V_\alpha = U_\alpha$ and we see immediately that this case is trivial, since $\rho_1 = 1 \in G$. For the inductive hypothesis, suppose that we have constructed a cover $\{U_{\alpha,k}\}$ such that $U_{\alpha,k} \subset U_\alpha$ and $\rho_\alpha : U_{\alpha,k} \to G$ satisfying $\phi_{\alpha\beta} = \rho_\alpha\varphi_{\alpha\beta}\rho_\beta^{-1}$ on $U_{\alpha,k} \cap U_{\beta,k}$ for $1 \le \alpha \le k$ and $1 \le \beta \le k$. Furthermore, assume that $M \subset \left(\bigcup\limits_{\alpha\le k} U_{\alpha,k}\right)\bigcup\left(\bigcup\limits_{\alpha>k} U_\alpha\right)$ and $|\exp^{-1}\rho_\alpha| \le c_k m$. We now claim that if m is sufficiently small, then we may continue this construction from $j = k$ to $j = k+1$, which will complete the inductive step and thus complete the proof.

For $j = k+1$, we define $u_j = \exp^{-1}(\phi_{j\beta}\rho_\alpha\varphi_{\alpha j})$ which defines a continuous map $u_j : U_{\alpha,k} \cap U_j \to \mathfrak{g}$ for $\alpha \le k = j-1$. If $m \le \frac{f_0}{c_k}$, we have $|\exp^{-1}\rho_\alpha(x)| \le c_k m \le f_0$ by the inductive hypothesis and $|\exp^{-1}(\phi_{j\alpha}\varphi_{\alpha j})| \le m \le f_0$ by assumption. The previous lemma then shows that u_j exists and satisfies the inequality

$$|u_j(x)| \le 2(1 + c_k)m = c_j m.$$

To see that u_j is consistently defined, on $U_j \cap \left(\bigcup_{\alpha \leq k} U_{\alpha,k} \right)$ we have by the consistency conditions of transition functions on a vector bundle that

$$\begin{aligned} u_j &= \exp^{-1}(\phi_{\alpha\beta}\rho_\alpha\varphi_{\alpha\beta}) \\ &= \exp^{-1}(\phi_{j\gamma}\phi_{\gamma\alpha}\rho_\alpha\varphi_{\alpha\gamma}\varphi_{\gamma j}) \\ &= \exp^{-1}(\phi_{j\alpha}\rho_\gamma\varphi_{\gamma j}), \end{aligned}$$

and so u_j is consistently defined.

Now choose a cutoff function ξ_j such that $\xi_j = 0$ on $U_j - \bigcup_{\alpha \leq k} U_{\alpha,k}$. This can be done in such a way such that the sets

$$U_{\alpha,j} = U_{\alpha,k} \cap \text{interior}\{x : \xi_j(x) = 1\}$$

cover $M - \overline{\bigcup_{\alpha > k} U_\alpha}$. If we then define $\rho_j = \exp(\xi_j u_j)$ on $U_j \cap \left(\bigcup_{\alpha \leq k} U_{\alpha,k} \right)$. Therefore $\rho_j = 1$ on $U_j - \bigcup_{\alpha \leq k} U_{\alpha,k}$ and $|\exp^{-1} \rho_j(x)| \leq |\xi_j(x)u_j(x)| \leq 2(1 + c_k)m = c_j m$. We see that the continuous map ρ_j and the sets $U_{\alpha,j}$ have the required properties for $j = k+1$, which completes the proof. $\square$

Remark 3.3.3. The condition (3.2) is subtle, although necessy for our cutoff function argument to make sense. By definition, $\rho_j = 1$ on $U_j - \bigcup_{\alpha \leq k} U_{\alpha,k}$, and so $\rho_j(x)$ must lie in the same co-ordinate patch as the identity for every $x \in M$, and it wouldn't necessarily if $\phi_{j\beta}\rho_\alpha\varphi_{\alpha j} \notin \tilde{G}$. The bound $\exp^{-1} |\phi_{j\beta}\varphi_{\alpha j}| \leq f_\ell$ for a suitable f_ℓ ensures this.

Remark 3.3.4. Since a bundle can be reconstructed from its transition functions, the importance of Proposition 3.3.2 will become clear when we construct sequences of transition functions in Lemma 3.3.7. Since each of these transition functions is different, although the covering of M is unchanged, it is our intenetion to be able to apply Proposition 3.3.2 to show that all the transition functions in the sequence do in fact describe E.

Corollary 3.3.5. *Let $\phi_{\alpha\beta}$ and $\varphi_{\alpha\beta}$ be two sets of $W^{2,p}$ transition functions on $U_\alpha \cap U_\beta$ for $2p > \dim M$, $\phi_{\alpha\beta} \in W^{2,p}(U_\alpha \cap U_\beta, G)$, $\varphi_{\alpha\beta} \in W^{2,p}(U_\alpha \cap U_\beta, G)$. Suppose*

$$m = \max_{\substack{(\alpha,\beta) \\ x \in U_\alpha \cap U_\beta}} |\exp^{-1} \phi_{\alpha\beta}(x)\varphi_{\alpha\beta}(x)| \leq f_\ell.$$

Then the ρ_α constructed in the previous proposition satisfy $\rho_\alpha \in W^{2,p}(V_\alpha, G)$. Furthermore, if $||\phi_{\alpha\beta}|_{U_\alpha \cap U_\beta}||_{W^{2,p}} \leq m'$ and $||\varphi_{\alpha\beta}|_{U_\alpha \cap U_\beta}||_{W^{2,p}} \leq m'$, then

$$||\exp^{-1} \rho_\alpha|_{V_\alpha}||_{W^{2,p}} \leq k(m').$$

Proof. For the range $2p > \dim M$, we see by the Sobolev multiplication, Lemma B.3 [57], that if $f, g \in W^{2,p}$, then $fg \in W^{2,p}$. We show the first part of the claim by induction. For $k = 1$, we see that $\rho_1 = 1 \in W^{2,p}(V_\alpha, G)$. Assume now that $\rho_\alpha \in W^{2,p}$ for $\alpha \leq k$, as in Proposition 3.3.2, then since

$2p > \dim M$, we have that $\phi_{j\beta}\rho_\alpha\varphi_{\alpha j} \in W^{2,p}$, and since $\exp^{-1}$ is smooth away from 0, $u_j \in W^{2,p}$. The cutoff function and exp are both smooth, so we have that $\rho_j \in W^{2,p}$, as claimed. The second claim follows immediate from the fact that $\xi_j u_j \in W^{2,p}$. □

We are now in a position to prove the main theorem of this section. Firstly, fix a $p > \frac{n}{2}$ and assume that $\int_M |F|^p * (1) \leq B$. The next Lemma proves the existence of a cover of M such that on each element of the cover the curvature of the connection is small enough for the local theorems to apply.

Lemma 3.3.6. *There exists a finite cover $\{U_\alpha\}_{\alpha\in I}$, $|I| < \infty$ of M depending on p, where $2p > n$ and B such that $B^n \simeq U_\alpha$ for each $\alpha \in I$. Under this co-ordinate identification we have*

$$\int_{|x|\leq B^n} |F_A|^p * (1) \leq \kappa'(E).$$

Proof. Choose a ball about each point $x \in M$ such that the curvature in this ball is restricted to $\int_{B^n} |F|^p * (1) \leq \kappa'(n)$. By the same construction and dilation as in Claim 3.2.5 in the proof of Theorem 3.2.1, we may assume that every $x \in M$ lies in a ball such that $\int_{|x|\leq B^n} |F|^p * (1) \leq \kappa'(E)$. Since M is compact, there exists a finite subcover of these patches which cover M. This choice of subcover is independent of D, but depends on B, κ' and the fact that $2p > n$. □

We can now combine the local theorem Theorem 3.2.1 with Proposition 3.3.2 and Lemma 3.3.6 to get the existence of a globally defined gauge which satisfies the Coulomb gauge condition when written in local co-ordinates. Firstly, set $\kappa' \leq \kappa$, where κ is that of Theorem 3.2.1 and then apply Theorem 3.2.1 to each of the elements U_α of the cover of M as constructed in Lemma 3.3.6. Recall now that choosing a gauge on a principal bundle is equivalent to choosing a trivialisation of the bundle (Lemma 2.2.8), and that a trivialisation of the principal bundle induces trivialisations of an associated vector bundle. As we showed in Chapter 2, choosing a gauge on our vector bundle is equivalent to choosing a gauge on its frame bundle, which in turn determines a trivialisation of the frame bundle and induces a trivialisation on the vector bundle. Such interplay is an excellent example of how the properties of the frame bundle as a principal bundle will be implicitly used without making mention of it. We have then that by choosing a gauge we have chosen a trivialisation $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^\ell$. In the following Lemma we bring together ideas which have been used already on individual connections (and overlap functions) and apply them to sequences.

Lemma 3.3.7. *Let $D(i)$ be a sequence of connections in $\mathcal{A}^{1,p}$ and assume that $\int_M |F_{D(i)}|^p * (1) \leq B$ for each i. Then there exists a fixed open cover U_α of M and trivialisations $\psi_\alpha(i) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^\ell$ which induces the connection forms $\psi_\alpha(i)(D(i)|_{U_\alpha})\psi_\alpha^{-1}(i) = d + A(i,\alpha)$. These trivialisations satisfy the properties*

(i) Conditions $(i) - (iv)$ of 3.2.1 are satisfied by the $A = A(i,\alpha)$ on U_α.

(ii) *The overlap functions* $\varphi_{\alpha\beta}(i) = \psi_\alpha(i) \circ \psi_\beta^{-1}(i)$ *are uniformly bounded in* $W^{2,p}(U_\alpha \cap U_\beta, G)$.

(iii) *For a subsequence, we have weak convergence*

$$A(i', \alpha) \rightharpoonup A(\alpha) \text{ in } W^{1,p}(U_\alpha, T^*M \otimes \mathfrak{g})$$
$$\varphi_{\alpha\beta}(i') \rightharpoonup \varphi_{\alpha\beta}(\infty) \text{ in } W^{2,p}(U_\alpha \cap U_\beta, G).$$

(iv) *The* $A(\alpha)$ *represent a connection* $D = d + A$ *on* E *presented in terms of a trivialisation of* E *whose overlap functions are given by* $\varphi_{\alpha\beta}(\infty)$.

Proof. Construct the cover of M as in Lemma 3.3.6 and note that by Rellich's lemma that $L^p \hookrightarrow L^{\frac{n}{2}}$ is a compact embedding for $p > \frac{n}{2}$, so $||F||_{L^p(U_\alpha)} \leq \kappa' \implies ||F||_{L^{\frac{n}{2}}(U_\alpha)} \leq C\kappa'$, and so the hypothesis of Theorem 3.2.1 are satisfied by each element of the cover for κ' small enough. We may then apply 3.2.1 to the connections to see that condition (i) is true.

Since, on $U_\alpha \cap U_\beta$, we have $A_\alpha = \varphi_{\alpha\beta} d\varphi_{\beta\alpha} + \varphi_{\alpha\beta} A_\beta \varphi_{\beta\alpha}$, we deduce that $\varphi_{\alpha\beta} \in W^{2,p}(U_\alpha \cap U_\beta, G)$ in exactly the same was as we deduce that $s \in W^{2,p}(B^n, G)$ in Lemma 2.10.2.

Since the Sobolev spaces $W^{1,p}(U_\alpha, T^*M \otimes \mathfrak{g})$ and $W^{2,p}(U_\alpha \cap U_\beta, G)$ are reflexive, a consequence of the Banach Alaoglu theorem states that any bounded sequence in these spaces has a weakly convergent subsequence, and (iii) follows.

The consistency conditions are preserved under weak limits, and so it is clear that $A(\alpha)$ is a connection on a bundle presented in terms of $\varphi_{\alpha\beta}(\infty)$. Although sequences of transition functions can coverge weakly so that the limit is over a different bundle. Note, however, that if we take $\varphi(\infty) = \varphi$, and $\varphi(i) = \phi$, where $\varphi(i)$ is any term in the sequence, then thanks to the uniform bounding of the transition functions, we can see that these satisfy the hypothesis of Proposition 3.3.2, and therefore these transition functions describe the same bundle, namely, E. □

We are now in a position to prove the main theorem.

Theorem 3.3.8 (Weak Uhlenbeck Compactness). *Let* $2p > \dim M$ *and* $D(i)$ *be a sequence of connections in* $\mathcal{A}^{1,p}$ *such that* $\int_M |F_{D(i)}|^p {*1} \leq B$. *Then there exists a subsequence* $\{i'\} \subset \{i\}$ *and gauge transformations* $S(i) \in \mathcal{G}^{2,p} = W^{2,p}(M, Aut\, E)$ *such that*

$$S(i')^{-1} \circ D(i') \circ S(i') \rightharpoonup D \text{ in } \mathcal{A}^{1,p}$$

Proof. Assume that the situation as described in 3.3.7 has been constructed and relabel $i' = i$. By the Sobolev embedding theorem for $2p > n$ we get that $W^{2,p}(M) \hookrightarrow C^0(M)$ is a compact embedding, so $\varphi_{\alpha\beta}(i) \to \varphi_{\alpha\beta}(\infty)$ strongly in $C^0(U_\alpha \cap U_\beta, G)$. Therefore there exists a fixed j, such that for $i < j \leq \infty$ we may apply Proposition 3.3.2 and Corollary 3.3.5 to $\varphi_{\alpha\beta}(i) = \phi_{\alpha\beta}$ and $\varphi_{\alpha\beta}(j) = \varphi_{\alpha\beta}$.

There exists a cover of M by the open sets $V_\alpha \subset U_\alpha$ such that for $j < i \leq \infty$, $\rho_\alpha(i) \in W^{2,p}(V_\alpha, G)$ and

$$\varphi_{\alpha\beta}(i) = \rho_\alpha(i)\varphi_{\alpha\beta}(j)\rho_\beta^{-1}(i)$$

Moreover, by construction, $\rho_\alpha \in W^{2,p}(V_\alpha, G)$ is bounded uniformly. Therefore, by the Banach-Alaoglu theorem there exists a subsequence which converges weakly to $\rho_\alpha(\infty)$. Again, by the compact embedding of $W^{2,p} \hookrightarrow C^0$ for the range $2p > n$ we also have the strong covergence $\rho_\alpha(i) \to \rho_\alpha(\infty)$.

Define the global gauge transformation $S(i) \in \mathcal{G}^{2,p}$ on U_α by the formula

$$S(i) = \psi_\alpha^{-1}(i)\rho_\alpha(i)\psi_\alpha(j) \tag{3.3}$$

On $U_\alpha \cap U_\beta$, the consistency condition

$$\psi_\alpha(i)\rho_\alpha(i)\psi_\alpha^{-1}(j) = \psi_\beta(i)\rho_\beta(i)\psi_\beta^{-1}(j)$$

can be rearranged to give

$$\rho_\alpha(i)\psi_\alpha(j)\psi_\beta(j)^{-1}\rho_\beta(i)^{-1} = \psi_\alpha(i)\psi_\beta^{-1}(i)$$

From the definition of the overlap functions, we see that

$$\rho_\alpha(i)\varphi_{\alpha\beta}(j)\rho_\beta(i)^{-1} = \varphi_{\alpha\beta}(i)$$

To show that $S(i)^{-1} \circ D(i) \circ S(i)$ is weakly convergent, fix a trivialisation $\psi_\alpha(j) : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^\ell$. This trivialisation does lie in $W^{2,p}$ since the transition functions do, although since $\mathcal{A}^{1,p}$ is an affine space, there is no natural choice of norm. It suffices to show that the induced connection forms in this trivialisation over V_α converge weakly in $W^{2,p}(V_\alpha, T^*M \otimes \mathfrak{g})$. From (3.3), we see that

$$\psi_\alpha(j)s(i)^{-1} \circ D(i) \circ s(i)\psi_\alpha^{-1}(j) = \rho_\alpha^{-1}(i)\psi_\alpha(i) \circ D(i) \circ \psi_\alpha^{-1}(i)\rho_\alpha(i)$$

Recall that the trivialisations $\psi_\alpha(i)$ were chosen such that $\psi_\alpha(i) \circ D(i) \circ \psi_\alpha^{-1}(i) = d + A(\alpha, i)$ satisfies (iii) of Lemma 3.3.7. Therefore, $S(i)^{-1} \circ D(i) \circ S(i)$ is now

$$\rho_\alpha^{-1}(i) \circ (d + A(\alpha, i)) \circ \rho_\alpha(i) = d + \rho_\alpha^{-1}d\rho_\alpha(i) + \rho_\alpha^{-1}(i)A(\alpha, i)\rho_\alpha(i)$$

Because $A(\alpha, i)$ is weakly convergent in $W^{1,p}(V_\alpha, T^*M \otimes \mathfrak{g})$ and $\rho_\alpha(i)$ in $W^{2,p}(V_\alpha, G)$ by Corollary 3.3.5, this connection converges weakly in $W^{1,p}(V_\alpha, T^*M \otimes \mathfrak{g})$. □

In Yang–Mills theory, one often wants to study the *moduli space* of connections, $\mathcal{M} = \mathcal{A}/\mathcal{G}$. The compactness result proven in Theorem 3.3.8 asserts that every subset of $\mathcal{A}^{1,p}(E)/\mathcal{G}^{2,p}(E)$ which satisfies an L^p bound on curvature is weakly compact, and this result has been fundamental in the developments of geometry and topology in the past 30 years. Unfortunately, any further discussion of the moduli space of connections would lead us too far astray, and so we refer to [15] for analysis of the moduli space and its significance.

Chapter 4

Removable Singularities in Yang–Mills Fields

4.1 Motivation

In this chapter, we specialise our attention to $n = 4$, which is both the dimension of most physical interest, and also the critical dimension for the Yang–Mills functional. As before, let E be a vector bundle of rank n over M, a smooth, orientable, compact and boundary-free Riemannian Manifold. As we know, the Yang–Mills functional is invariant under gauge transformations, and it is this invariance that is the main problem when one considers the regularity theory. For example, suppose that $A \in \Omega^1(\text{ad } E)$ is a Yang–Mills connection form so that F_A is a smooth Yang–Mills field. If $S \in \mathcal{G}$ is an element of the gauge group which is *not necessarily smooth*, then $\tilde{A} = S^{-1}dS + S^{-1}AS$ is also a Yang–Mills connection, and $F_{\tilde{A}}$ is a Yang–Mills field; however, there is no reason for $\tilde{A}$ or $F_{\tilde{A}}$ to be smooth, or even continuous, and such a transformation might even introduce apparent *point singularities*. In her paper 'Removable Singularities in Yang–Mills Fields', [54], Uhlenbeck prescribes conditions under which it is possible to identify when there is one of these 'removable singularities', and then provides a method for removing it. Much of her argument is still valid for dimensions not equal to 4, but there are several crucial theorems which hinge on the dimension. It is Tao and Tian's paper, [50], which gives a description of removable singularities in higher dimensions, although we only consider Uhlenbeck's paper in this chapter.

It should first be reiterated that it is very much possible to have singularities in Yang–Mills fields, and it will not always be possible to remove them. For example, let D be a Yang–Mills connection in a vector bundle E over S^{n-1} and let $f : B^n - \{0\} \to S^{n-1}$ be given by $f(x) = \dfrac{x}{|x|}$. The pullback connection f^*D is then a Yang–Mills connection on the pullback bundle f^*E over B^n and the curvature grows *exactly* like $\dfrac{1}{|x|^2}$. It is known that there exist non trivial Yang–Mills fields over S^2, S^3, S^4 (e.g. in the Hopf bundle), and this gives us some examples of isolated singularities in dimensions 3,4 and 5. Moreover, since the curvature grows exactly like $\dfrac{1}{|x|^2}$ the integral $\displaystyle\int_{B^n} |f^*(F)|^q * (1)$ is finite for $q < \frac{n}{2}$, but infinite for $q \geq \frac{n}{2}$ because the Jacobian in spherical co-ordinates will dominate the singularity for $q < \frac{n}{2}$, otherwise the singularity dominates and the integral blows up.

This fact will be important in the identification of removable singularities as opposed to inherent singularities.

4.2 Canonical Choice of Gauge

Before we get to removing singularities we must first introduce some notation. In this section we follow Uhlenbeck's construction of the canonical choice of gauge over three particular regions of interest. Namely, we construct controlled gauges in a domain U, where $D = d + A$ and A satisfies the Coulomb condition, $d^*A = 0$, when $||F||_{L^\infty}$ is small enough. The three regions of interest for us will be the unit sphere, the unit ball and the annulus, i.e.

$$\begin{aligned} U = B^n &= \{x \in \mathbb{R}^n : \ |x| \leq 1\}, \\ U = S^{n-1} &= \{x \in \mathbb{R}^n : \ |x| = 1\}, \\ U = \mathfrak{A} &= \{x \in \mathbb{R}^n : \ 1 \leq x \leq 2\}. \end{aligned}$$

There are many methods to choose a local gauge; however, the most intuitive is to fix a fibre over x_0 and identify nearby fibres by setting $(x(t) \cdot A(x(t)) = 0$ along all geodesics $x(t)$ emanating from x_0. By doing this, we *almost* fix a gauge in all balls within the cut locus of M, in that the gauge is fixed up to a gauge change by a constant element of G, but more will be said on this later. In a Euclidean ball, this corresponds to $A(0) = 0$ and $A_r = 0$. In the physics literature, this is known as the Poincaré, or multipolar gauge condition, but we refer to it by its original mathematical name as an exponential gauge.

To simplify our notation, we use the coordinate change $x = (x_i)_{i=1}^n \cong (r, \psi) = (|x|, \frac{x^i}{|x|})$ for $\psi = \dfrac{x}{|x|} \in S^{n-1}$ as a transformation from Euclidean to spherical coordinates. The 1-form $A = (A_i) \cong (A_r, A_\psi)$ splits into radial and spherical parts. The two form $F = (F_{ij}) \cong (F_{r\psi}, F_{\psi\psi})$ splits into two parts also. Note that $F_{rr} = 0$ because of anti-symmetry. Here $F_{\psi\psi}$ is a two-form along S^{n-1}. In the sphere S^{n-1}, we often change coordinates to $\psi \cong (\varphi, \theta)$, i.e. from spherical to polar coordinates. Here $\varphi \in (0, \pi)$ and $\theta \in S^{n-2}$. Therefore, on S^{n-1}, $A = (A_\psi) \cong (A_\varphi, A_\theta)$ and $F_A = (F_{\varphi\varphi}) \cong (F_{\varphi\theta}, F_{\theta\theta})$ ($F_{\varphi\varphi} = 0$ by antisymmetry). Furthermore, although the local identification $D = d + A =: d_A$ is only valid on co-ordinate patches, as noted in Section 2.9, by an abuse of notation we will swap between writing d_A and D to mean a connection, since we are assuming that in every co-ordinate patch this identification is valid. We will adopt a similar approach for the curvature, and by a further abuse of notation swap between F_A and F. When writing the curvature in local co-ordinates, the reference to the connection form A will be surpressed if it is understood what the connection form is, and we will simply write F_{ij}. Similarly, if the emphasis is on the co-ordinate system, then we will write $F(x)$, $F(y)$ etc without reference to the connection form in order to emphasise the local co-ordinates.

Lemma 4.2.1. *In an exponential gauge in* $\mathbb{R}^n$,

$$|A(x)| \leq \frac{1}{2}|x| \max_{|y|\leq|x|} |F(y)|. \tag{4.1}$$

Proof. To start, assume that a gauge is given in which $d_{\tilde{A}} = d + \tilde{A}$. Then we would like to find a gauge transformation such that $A_r(x) = S^{-1}\frac{\partial S}{\partial r} + S^{-1}\tilde{A}_r S = 0$. Therefore, we require $\frac{\partial S}{\partial r} = -\tilde{A}_r S$. If $S(x) = \sigma(|x|, \frac{x}{|x|})$, then we see that this is equivalent to the ODE with ψ fixed

$$\frac{d}{dt}\sigma(t,\psi) = -\tilde{A}_r(t,\psi)\sigma(t,\psi)$$

with the initial condition $\sigma(0,\psi) = 1 \in G \subset SO(\ell)$. If $\tilde{A}$ is smooth, then $S(x) = \sigma(|x|, \frac{x}{|x|}), S^{-1}\frac{\partial S}{\partial r} \in C^1(B^n, G)$. By construction, if $d_A = d + A$, where $A = S^{-1}dS + S^{-1}\tilde{A}S$, then $\sum_{k=1}^{n} x^k A_k(x) = A_r(x) = 0$, which is a Neumann-type condition. Note that although $S(x) \in C^1(B^n, G)$, it is not necessary that A is as smooth as $\tilde{A}$, or that $F_A = S^{-1}F_{\tilde{A}}S$, although both exist. We compute

$$\sum_k x^k F_{kj} = \sum_k \Big(x^k \frac{\partial A_j}{\partial x^k} - x^k \frac{\partial A_k}{\partial x^j} + x^k [A_k, A_j] \Big),$$

and since $\sum_k x^k A_k(x) = 0$,

$$\frac{\partial}{\partial x^j} \sum_k x^k A_k = A_j + \sum_k x^k \frac{\partial A_k}{\partial x^j} = 0,$$

together with $\frac{\partial}{\partial r} = \sum_k \frac{x^k}{r}\frac{\partial}{\partial x^k}$, we find that

$$\begin{aligned}\sum_k x^k F_{kj} &= r\frac{\partial A_j}{\partial r} + A_j \\ &= \frac{\partial}{\partial r}(rA_j).\end{aligned}$$

Note that $x^k[A_j, A_k] = 0$ by the Jacobi identity and the fact that $\sum_k x^k A_k(x) = 0$. We then integrate both sides to get

$$\begin{aligned}\int_0^{|x|} \frac{\partial}{\partial r}(rA_j(x))dr = |x|A_j(x) &= \int_0^{|x|} \sum_k x^k F_{kj}(x)dr \\ &= \max_{|y|\leq|x|} |F(y)| \int_0^{|x|} \left| \sum_k x^k dr \right| \\ &\leq \max_{|y|\leq|x|} |F(y)| \int_0^{|x|} |x| dr \\ &= \frac{|x|^2}{2} \max_{|y|\leq|x|} |F(y)|\end{aligned}$$

And so $|A_j(x)| \leq \frac{1}{2}|x| \max_{|y|\leq|x|} |F(y)|$, as advertised. □

The construction of an exponential gauge in the sphere is analogous; however, the estimates are slightly different due to the underlying curvature.

Lemma 4.2.2. *In an exponential gauge on the sphere S^{n-1},*

$$||A(\varphi,\theta)|| \leq \frac{1-\cos\varphi}{\sin\varphi}||F_A||_\infty$$

away from the cut locus.

Proof. Assume that we have constructed a gauge as in the previous lemma such that the radial component is 0. In this gauge, the 'radial' component is φ, so $A_\varphi(\varphi,\theta)=0$. Therefore, by the definition of curvature, we have

$$\begin{aligned} F_{\varphi\theta} &= \frac{\partial A_\theta}{\partial\varphi} - \frac{\partial A_\varphi}{\partial\theta} + [A_\varphi, A_\theta] = \frac{\partial A_\theta}{\partial\varphi} \\ \implies |A_\theta(\varphi,\theta)| &= \Big|\int_0^\varphi F_{\varphi\theta}(\tau,\theta)\mathrm{d}\tau\Big|. \end{aligned}$$

Since we are on the sphere, the metric is no longer flat, and so the norm is not just the absolute value anymore, but rather

$$\begin{aligned} ||A_\theta(\varphi,\theta)|| &= (\sin\varphi)^{-1}|A_\theta(\varphi,\theta)| \\ &= (\sin\varphi)^{-1}\Big|\int_0^\varphi F_{\varphi\theta}(\tau,\theta)\frac{\sin\tau}{\sin\tau}\mathrm{d}\tau\Big| \\ &\leq \frac{\int_0^\varphi \sin\tau\mathrm{d}\tau}{\sin\varphi}\max_{\phi\in[0,\pi)}\big|F_{\phi\theta}(\phi,\theta)\frac{1}{\sin\phi}\big| \\ &= \frac{1-\cos\varphi}{\sin\varphi}\max_{\phi\in[0,\pi)}\big|F_{\phi\theta}(\phi,\theta)\frac{1}{\sin\phi}\big|. \end{aligned}$$

Therefore, by the identity $\dfrac{1-\cos\varphi}{\sin\varphi} = \tan\dfrac{\varphi}{2}$, we have the estimate

$$||A_\theta(\varphi,\theta)|| \leq \tan\frac{\varphi}{2}||F_A||_{L^\infty}.$$

At the cut locus from $\varphi = 0$, as $\varphi\to\pi$, the estimate blows up and the exponential gauge becomes singular. □

Lastly, suppose we have a gauge which has been given on $E|_{S^{n-1}} \simeq S^{n-1}\times\mathbb{R}^\ell$. Then we may extend it to a collar neighbourhood with $A_r = 0$. Analogously to Lemma 4.2.1, we have

$$\int_{\frac{1}{|x|}}^1 \frac{\partial}{\partial r}(rA_j(x))dr = \left(1-\frac{1}{|x|}\right)A_j(x) = \int_{\frac{1}{|x|}}^1 \sum_k x^k F_{kj}(x)dr.$$

Note that this fixes the gauge on the boundary. From this, we get the estimate

$$|A(x)| \leq \frac{1}{|x|}\left|A_\psi\left(\frac{x}{|x|}\right)\right| + \left(|x| + \frac{1}{|x|}\right)\max_{|y|\leq|x|\leq 1}|F(y)|. \tag{4.2}$$

In the next three lemmas, we match these constructed gauges on overlapping regions such that a gauge can be defined over the whole region. The argument for each construction is very similar; however, due to the boundary conditions, we state them separately.

Lemma 4.2.3. *There exists $\alpha_0 > 0$ and $\kappa < \infty$ depending on G, such that if D is a connection in a bundle over S^{n-1} in which*

$$\max_{\psi \in S^{n-1}} |F_A| = ||F_A||_{L^\infty} < \alpha_0$$

then there exists a connection $d_{\tilde{A}} = d + \tilde{A}$ such that $||A||_{L^\infty} \leq \kappa ||F_A||_{L^\infty}$.

Proof. By Lemma 4.2.3, there exists an exponential gauge off the north pole ($\varphi = 0$) such that $d_{A_0} = d + A^0$, and an exponential gauge off the south pole ($\varphi = \pi$) such that $d_{A^\pi} = d + A^\pi$, and that these gauges match in regions where they are both defined. From Lemma 4.2.3, we have

$$||A^0(\varphi,\theta)|| = \csc\varphi |A^0(\varphi,\theta)| \leq \frac{\varphi}{2} ||F_A||_{L^\infty}$$
$$||A^\pi(\varphi,\theta)|| = \csc\varphi |A^\pi(\varphi,\theta)| \leq \frac{\pi - \varphi}{2} ||F_A||_{L^\infty}$$

In the region about $\varphi = \frac{\pi}{2}$ we have $d + A^0 = d + A^\pi$, and so we have $d + A^\pi - A^0 = d$, and so there exists a gauge transform such that $A^\pi - A^0 = S^{-1}dS$. Since these are exponential gauges, we have $A^\pi_\varphi = A^0_\varphi = 0$, and so $\frac{\partial S}{\partial \varphi} = 0$. Therefore $S(\varphi,\theta)$ is independent of φ and we let $\tilde{S}(\theta) = s(\varphi,\theta)$ for notational convenience. Moreover, by hypothesis and since s is independent of φ, we have

$$|d\tilde{S}(\theta)| = |dS(\frac{\varphi}{2},\theta)| = |A^0_\theta(\frac{\varphi}{2},\theta) - A^\pi_\theta(\frac{\varphi}{2},\theta)| \leq 2||F_A||_{L^\infty} \leq 2\alpha_0,$$

where α_0 is half of the injectivity radius of the Lie group, G. Then we may define $\tilde{S}$ using the exponential map, such that

$$\tilde{S}(\theta) = S_0 \exp(u(\theta)),$$

where $u : S^{n-2} \to \mathfrak{g}$. If we assume that $\displaystyle\int_{S^{n-1}} u = 0$ so that $u \equiv C \implies u \equiv 0$, then we have that

$$||du||_{L^\infty} \leq C(G)||d\tilde{S}||_{L^\infty} \leq 2C(G)||F_A||_{L^\infty},$$

where $C(G)$ depends only on the Lie group, or, more specifically, its injectivity radius. Then, we define a new gauge by multiplying the exponential gauge from the north pole by $h : S^{n-1} - \{0,\theta\} \to G$

$$h(\varphi,\theta) = S_0 \exp\Big(\sin^2(\frac{\varphi}{2}) u(\theta)\Big).$$

By construction, this is exactly the same gauge defined by changing the exponential gauge from the south by $q : S^{n-1} \setminus \{\pi,\theta\} \to G$ given by

$$q(\varphi,\theta) = S_0 \exp\Big(-\cos^2(\frac{\varphi}{2}) y(\theta)\Big).$$

This new gauge is then globally defined on S^{n-1}, where

$$A = \begin{cases} h^{-1}A^0 h + h^{-1}dh & \text{for } 0 \leq \varphi \leq \frac{\pi}{2}, \\ q^{-1}A^\pi q + q^{-1}dq & \text{for } \frac{\pi}{2} \leq \varphi \leq \pi. \end{cases}$$

We can see that this is continously defined, since for $\varphi = \frac{\pi}{2}$, we have $h^{-1}A^0h + h^{-1}dh = q^{-1}A^\pi q + q^{-1}dq$. Since $A^0_\varphi = A^\pi_\varphi = 0$, we have that

$$\begin{aligned} |A_\varphi(\varphi,\theta)| &= \begin{cases} |h^{-1}\frac{\partial}{\partial\varphi}h| & \text{for } 0 \leq \varphi \leq \frac{\pi}{2} \\ |q^{-1}\frac{\partial}{\partial\varphi}q| & \text{for } 0 \leq \varphi \leq \frac{\pi}{2} \end{cases} \\ &= |\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}u(\theta)| \\ &\leq \frac{1}{2}\kappa||F_A||_{L^\infty}. \end{aligned}$$

Since $||A_\theta(\varphi,\theta)|| = \csc\varphi|A_\theta(\varphi,\theta)|$, we have

$$\begin{aligned} \csc\varphi|A_\theta(\varphi,\theta)| &\leq \begin{cases} \csc\left(|A^0(\varphi,\theta)| + |\frac{\partial}{\partial\theta}h|\right) & \text{for } 0 \leq \varphi \leq \frac{\pi}{2} \\ \csc\left(|A^\pi(\varphi,\theta)| + |\frac{\partial}{\partial\theta}q|\right) & \text{for } \frac{\pi}{2} \leq \varphi \leq \pi \end{cases} \\ &\leq \frac{1}{2}\kappa\csc\varphi||F_A||_{L^\infty}. \end{aligned}$$

Therefore

$$||A||_{L^\infty} \leq \kappa||F_A||_{L^\infty},$$

as claimed. □

It is in the next two proofs where it will be noticable that we have only fixed a gauge up to a constant gauge transformation, although it will not yet be important. We aim to construct a gauge over the unit ball and an annulus where the gauge on the boundary is prescribed, as in Dirichlet boundary conditions; however, due to the extra degree of freedom this makes these boundary conditions confusing.

Lemma 4.2.4. *Let E be a bundle over B^n with covariant derivative $d_{\tilde{A}} = d + \tilde{A}$ and curvature $||F_{\tilde{A}}||_{L^\infty} \leq \alpha$. Assume a gauge is fixed on $E|_{S^{n-1}} = E|_{\partial B^n}$ in which $d_{\tilde{A}_\psi} = d + \tilde{A}_\psi$, $|\tilde{A}(1,\psi)| \leq \alpha$ for all $\psi \in S^{n-1}$. Then there exists $\alpha_1 = \alpha_1(G)$ such that if $\alpha < \alpha_1$, there exists a gauge on $E|_{B^n}$ in which $d_A = d + A$, $\tilde{A}_\psi = A_\psi$ on $E|_{S^{n-1}}$ and $||A||_{L^\infty} \leq \kappa\alpha$.*

Proof. Similarly to the previous lemma, this is an exercise in gauge matching, although in this case we match an exponential gauge in $\mathbb{R}^n$ to an exponential gauge off the sphere. Let $d_{A^0} = d + A^0$ be the exponential gauge from zero and match this by rotation with $d_{A^1} = d + A^1$, the exponential gauge off the sphere. This fixes $\tilde{A}_\psi$. From (4.1), we have

$$|A^0(x)| \leq \frac{1}{2}|x|||F_{\tilde{A}}||_{L^\infty} \leq \frac{1}{2}|x|\alpha.$$

Similarly, from (4.2), we have

$$|A^1(x)| \leq \frac{1}{|x|}||\tilde{A}_\psi|| + \left(|x| + \frac{1}{|x|}\right)||F_{\tilde{A}}||_{L^\infty} \leq \frac{3}{|x|}\alpha.$$

Analogously to the previous lemma, we know that the two gauges are related by a gauge change $s(\psi)$, and that for α small enough, we have $S = S_0 \exp(\tilde{u}(\psi))$, where $\tilde{u} : S^{n-2} \to \mathfrak{g}$. Due to the singularity at the origin, we change the gauge from the origin by $\tilde{S}(r,\psi) = S_0 \exp\big(r^2\tilde{u}(\psi)\big)$. Then,

$$A^1 = \tilde{S}^{-1}d\tilde{S} + \tilde{S}A^0\tilde{S} \qquad \text{on } S^{n-1}$$

by construction, and the gauge given by

$$A = \tilde{S}^{-1}d\tilde{S} + \tilde{S}A^0\tilde{S} \qquad \text{on } B^n$$

satisfies

$$||A||_{L^\infty} \leq \kappa\alpha.$$

□

Lemma 4.2.5. *Let E be a bundle with covariant derivative d_A over $\mathfrak{A} = \{x : 1 \leq |x| \leq 2\}$ and curvature $||F_A||_{L^\infty} \leq \alpha$. Let $S_t^{n-1} = \{x : |x| = t\}$. Suppose gauges are chosen on $E|_{S_t^{n-1}}$ in which $d_{\tilde{A}^t_\psi} = d + \tilde{A}^t_\psi$ with $|\tilde{A}^t_\psi(t,\psi)| \leq \alpha$ for $t = \{1,2\}$. Then there exists $\alpha_2 > 0$ such that for $\alpha < \alpha_2$, there is a gauge on $E|_{\mathfrak{A}}$ in which $d_A = d + A$, $\tilde{A}^t_\psi = A_\psi$ on S_t^{n-1} for $t = \{1,2\}$, and $||A||_{L^\infty} \leq \kappa\alpha$.*

Proof. Construct exponential gauges off S_t^{n-1} for $t = \{1,2\}$ and then match them on the sphere S_1^{n-1}. Since $d_A = d+\tilde{A}^1 = d+\tilde{A}^2$, $\tilde{A}^1$ and $\tilde{A}^2$ are related by a gauge change. Therefore

$$S^{-1}dS = \tilde{A}^1 - \tilde{A}^2.$$

Analogously to the previous lemmas, since $\tilde{A}^1$ and $\tilde{A}^2$ are in exponential gauge, we find that s is independent of r, and that $S = S_0 \exp \hat{u}(\psi)$, where $\hat{u} : S^{n-2} \to \mathfrak{g}$, since $|\tilde{A}^t_\psi(1,\psi)| \leq \alpha$. Then, change the gauge off S_2^{n-1} by $\hat{S}(r,\psi) = S_0 \exp\big((2-r)^2\hat{u}(\psi)\big)$, and the result follows. □

Now that we have constructed gauges on our regions of interest, we would like to prove that there exists a Coulomb gauge representative for each one. In sight of the method of doing this in Chapter 1, we would like to use the implicit function theorem to solve

$$d^*(S^{-1}dS + S^{-1}\tilde{A}S) = d^*A = 0,$$

for s when $\tilde{A}$ is sufficiently small. Unfortunately, we have only constructed $A \in L^\infty(M, \text{ad } E \otimes T^*M)$, and you can only 'trade' derivatives for higher integrability in the Sobolev embedding theorem, not the other way around. As such, we are not able to use the map $W^{2,p} \to L^p$, since we are not guaranteed that $S \in W^{2,p}(M, \text{Aut } E)$, but only that $S \in W^{1,p}(M, \text{Aut } E)$. As such, we must use the map $W^{1,p} \to W^{-1,p}$, where $W^{-1,p}$ is the dual space to $W^{1,q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.2.6. *Let E be a bundle over S^{n-1} with covariant derivative $d_{\tilde{A}} = d+\tilde{A}$ and curvature $F_{\tilde{A}}$. Then there exists $\gamma_0 > 0$ such that if $||F_{\tilde{A}}||_{L^\infty} \leq \gamma_0$, then there exists a gauge in which $d_A = d + A$ and $d^*A = 0$. Furthermore, $||A||_{L^\infty} \leq K||F_A||_{L^\infty}$. The choice of gauge is unique up to constant multiplication by an element of G.*

Proof. For $\gamma_0 \leq \alpha_0$, by Lemma 4.2.3 we may construct a gauge $d_A = d + \tilde{A}$ such that $||\tilde{A}||_{L^\infty} \leq \kappa ||F_{\tilde{A}}||_{L^\infty} \leq \kappa\gamma_0$. For any $\infty > p > n$ the expression

$$Q(u, B) = d^*[\exp(-u)d\exp(u) + \exp(-u)Bu]$$

induces a smooth map

$$Q : W^{1,p}(S^{n-1}, \mathfrak{g}) \times L^p(S^{n-1}, \mathfrak{g} \otimes T^*M) \to W^{-1,p}(S^{n-1}, \mathfrak{g})$$

as in Chapter 1. By Stokes' Theorem we have that the image lies in

$$W_\perp^{-1,p}(S^{n-1}, \mathfrak{g}) = \{\xi \in W^{-1,p}(S^{n-1}, \mathfrak{g}) : \langle \xi, u_0 \rangle = 0, u_0 \in \mathfrak{g}\}.$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing. Note that although $W^{-1,p}$ is *not* the dual of $W^{1,p}$ unless $p = 2$, we may consider the elements of $\mathfrak{g}$ as the constant elements of $W^{1,p}$, which also lie in $W^{1,q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, and so the dual pairing makes sense. Likewise, we define

$$W_\perp^{1,p}(S^{n-1}, \mathfrak{g}) = \left\{ u \in W^{1,p}(S^{n-1}, \mathfrak{g}) : \int_{S^{n-1}} u = 0 \right\}$$

to ensure that the if $u \equiv C$, then $u \equiv 0$. The linearisation of Q is then

$$dQ = \begin{bmatrix} \frac{d}{dt} Q(u + tv, B)|_{t=0} \\ \frac{d}{dt} Q(u, B + tv)|_{t=0} \end{bmatrix}.$$

Explicity, for $d_1 Q_{(0,0)} : W_\perp^{1,p}(S^{n-1}, \mathfrak{g}) \to W_\perp^{-1,p}(S^{n-1}, \mathfrak{g})$, we have

$$\begin{aligned} d_1 Q_{(0,0)} &= \frac{d}{dt} Q(u + tv, B)|_{t,u,B=0} \\ &= \frac{d}{dt} d^*(e^{-u-tv} de^{u+tv} + e^{-u-tv} Be^{u+tv})_{u,B=0} \\ &= d^*(-ve^{-u}de^u + e^{-u}d(ve^u) - ve^{-u}Be^u + e^{-u}Bve^u)|_{u,B=0} \\ &= (d^*dv + [B, v])|_{B=0} \\ &= d^*dv = \Delta v. \end{aligned}$$

Note that this map is injective since the only elements in the kernel of Δ in $W^{1,p}(S^{n-1}, \mathfrak{g})$ are constant functions, but the orthonality condition implies that the only constant function in $W_\perp^{1,p}(S^{n-1}, \mathfrak{g})$ is zero, and so Δ is injective. Moreover, the map is surjective since for every $g \in L^p$ there exists an $f \in W^{1,p}$ such that $-\Delta f = g$ weakly by a standard existence argument for Poisson's equation - see, for example [52] Chapter 8. We see then that this is a Banach space isomorphism. The implicit function theorem for Banach spaces then establishes the existence of a solution to

$$Q(u, \tilde{A}) = d^*(S^{-1}dS + S^{-1}\tilde{A}S) = d^*A = 0,$$

provided $\tilde{A} \in L^p(S^{n-1}, \mathfrak{g} \otimes T^*M)$ is sufficiently small, where $S = \exp u \in W^{1,p}(S^{n-1}, G)$ and $u \in W_\perp^{1,p}(S^{n-1}, \mathfrak{g})$. This is always possible, since we may take γ_0 small such that $||\tilde{A}||_{L^p} \leq C||\tilde{A}||_{L^\infty} \leq C\kappa\gamma_0$. By construction, the norm $||u||_{W^{1,p}}$ is also assumed to be small, and so we have $||A||_{L^p} \leq (1+\kappa)||\tilde{A}||_{L^\infty}$, where $A = S^{-1}dS + S^{-1}\tilde{A}S$. By Hodge theory, see, for instance, [57] Lemma 5.1, we have $||A||_{W^{1,q}} \leq C\big(||dA||_{L^p} + ||d^*A||_{L^p} + ||A||_{L^p}\big)$. Since

$|F_A| = |F_{\tilde{A}}|$, we have $F_A \in L^\infty$, and since $d^*A = 0$, $F_A = dA + A \wedge A$, we have the following estimate

$$||A||_{W^{1,q}} \leq C(||F_A||_{L^q} + ||A||^2_{L^{2q}} + ||A||_{L^q}).$$

Since we have $A \in L^p$, by setting $q = \frac{p}{2}$, we have $A \in W^{1,\frac{p}{2}}$. For $p \geq \frac{3n}{2}$, which we have, since $p > n$, we have then by then Sobolev embedding $W^{1,\frac{p}{2}} \hookrightarrow L^{2p}$ with the estimate $||A||_{L^{2p}} \leq C||A||_{W^{1,\frac{p}{2}}}$. Therefore, $A \in W^{1,p}$. Then, for $n \geq 2$, we have that $q = \frac{1}{2}\frac{np}{n-p} > p$, and that $W^{1,p} \hookrightarrow L^{2q}$, so we get that $A \in W^{1,q}$. Iteratively, we define $q_i = \frac{1}{2}\frac{nq_{i-1}}{n-q_{i-1}}$ with $q_0 = p$, so that $||A||_{L^{2q_i}} \leq C||A||_{W^{1,q_{i-1}}}$, and we have that $A \in W^{1,q_i}$. For $2q_i > n$ for some $i \in \mathbb{N}$, we have that $W^{1,q_i} \subset L^\infty$, and so we get the claimed estimate on $||A||_{L^\infty}$. □

Corollary 4.2.7. *Under the hypothesis of Theorem 4.2.6 with $n = 4$, we have*

$$(2 - K||F||_{L^\infty}) \int_{S^3} |A|^2 \leq \int_{S^3} |F_A|^2.$$

Proof. Since $d^*A = 0$ (A is co-closed) on S^3, we have

$$\lambda_1 \int_{S^3} |A|^2 \leq \int_{S^3} |F_A|^2,$$

where λ_1 is the first eigenvalue of the Laplace operator on co-closed one forms. By spectral analysis, see for instance [4], we have $\lambda_1 = 4$. Since $F_A = dA + A \wedge A$, we have

$$\begin{aligned}\Big(4\int_{S^3} |A|^2\Big)^{1/2} &\leq \Big(\int_{S^3} |dA|^2\Big)^{1/2} \leq \Big(\int_{S^3} |F_A|^2\Big)^{1/2} + \Big(\int_{S^3} |A|^4\Big)^{1/2} \\ &\leq \Big(\int_{S^3} |F_A|^2\Big)^{1/2} + K||F_A||_{L^\infty}\Big(\int_{S^3} |A|^2\Big)^{1/2}.\end{aligned}$$

□

Theorem 4.2.8. *Let D be a covariant derivative in a bundle over B^n. There exists $0 < \gamma_1 < \gamma_0$ such that if $||F||_{L^\infty} \leq \gamma_1$, then there exists a gauge for E over B^n such that if $D = d + A$ in this gauge, then $d^*A = 0$ in B^n and $d^*_\psi A_\psi = 0$ on S^{n-1}. Furthermore $||A||_{L^\infty} \leq \tilde{\kappa}_1||F_A||_{L^\infty}$.*

Remark 4.2.9. By the notation d_ψ, we mean that index of the exterior derivative only runs over the co-ordinates ψ, not r. This is clearly the case if one only considers the sphere, but we add the extra notation to specify that the exterior derivative of the radial and spherical parts of A must be in Coulomb gauge independently on the boundary spheres.

Proof of 4.2.8. Apply Theorem 4.2.6 to fix a gauge on $S^{n-1} = \partial B^n$ such that $d^*_\psi A_\psi = 0$. In the construction of this gauge on the sphere, we have $\{F_{ij}\} = \{F_{\psi\psi}\}$, and so $||A_\psi||_{L^\infty} \leq K||F_{\psi\psi}||_{L^\infty}$. Then, if $K\gamma_1 \leq \alpha_1$ and $\gamma_1 \leq \alpha_1$, by Lemma 4.2.4 we may construct a gauge over B^n such that if $d_{\tilde{A}} = d + \tilde{A}$ in this gauge, then $||\tilde{A}||_{L^\infty} \leq \kappa_1\gamma_1$. We then intend to solve the equation

$$Q(u, \tilde{A}) = d^*A = d^*(S^{-1}dS + S^{-1}\tilde{A}S) = 0$$

by the implicit function theorem. Here $s = \exp\ u$, where $u|_{S^{n-1}} = 0$. For $p > n$ the map

$$Q:\ W_0^{1,p}(B^n, \mathfrak{g}) \times L^p(B^n, \mathfrak{g} \otimes T^*M) \to W^{-1,p}(B^n, \mathfrak{g}),$$

is smooth. Then, since the Poisson equation with Dirichlet boundary conditions always has a unique weak solution, see for instance Theorem 8.3 of [52], the linearisation

$$d^1Q(0,0) = \Delta :\ W_0^{1,p}(B^n, \mathfrak{g}) \to W^{-1,p}(B^n, \mathfrak{g})$$

is an isomorphism. Since $||\tilde{A}||_{L^\infty} \leq \kappa_1\gamma_1$, we can make $||\tilde{A}||_{L^p}$ arbitrarily small by choosing γ_1 small enough. We then apply the implicit function theorem for Banach spaces to yield a solution, and the regularity of the solution follows exactly the same argument as Theorem 4.2.6. □

Theorem 4.2.10. *Let D be a covariant derivative in a bundle E over $\mathfrak{A} = \{x : 1 \leq |x| \leq 2\}$. There exists $\gamma' > 0$ such that if $||F_A||_\infty \leq \gamma'$, then there exists a gauge in which $d_A = d + A$, $d^*A = 0$ in $\mathfrak{A}$, $d^*_\psi A_\psi = 0$ on S_1^{n-1} and S_2^{n-1}, and $\int_{|x|=t} A_r = 0$ for all $t \in [1,2]$. Moreover, $||A||_{L^\infty} \leq K'||F_A||_{L^\infty}$.*

Proof of Theorem 4.2.10. Firstly, we apply Theorem 4.2.3 on the boundary spheres, S_t^{n-1}, $t \in \{1, 2\}$ and construct $\tilde{A}$ as in Lemma 4.2.5. Once again, we want to solve the equation

$$Q(u, \tilde{A}) = d^*A = d^*(S^{-1}dS + S^{-1}\tilde{A}S) = 0,$$

for $s = \exp\ u$; however, in order to preserve the condition that $d^*_\psi A_\psi = 0$, we enforce the boundary conditions $u = C$ on $\partial\mathfrak{A} = S_1^{n-1} \cup S_2^{n-1}$, where $C :\ \mathfrak{A} \to \mathfrak{g}$ is any constant map. This motivates the definition

$$\begin{aligned} W_\perp^{1,p}(\mathfrak{A}, \mathfrak{g}) = \{u \in W^{1,p}(\mathfrak{A}, \mathfrak{g}) :\ & u|_{S_t^{n-1}} = \text{const for } t \in \{1, 2\} \text{ and} \\ & u \text{ is } L^2 \text{ perpendicular to the constants of } \mathfrak{g}\}. \end{aligned}$$

As in the previous theorems, for $p > n$, Q induces a smooth map

$$Q:\ W_\perp^{1,p}(\mathfrak{A}, \mathfrak{g}) \times L^p(\mathfrak{A}, \mathfrak{g} \otimes T^*M) \to W^{-1,p}(\mathfrak{A}, \mathfrak{g}).$$

The linearisation $d^1Q(0,0) = \Delta :\ W_\perp^{1,p}(\mathfrak{A}, \mathfrak{g}) \to W^{-1,p}(\mathfrak{A}, \mathfrak{g})$ is injective by the same argument as in Theorem 4.2.6 not an isomorphism, since the solution is invariant under a rotation of a constant element of $\mathfrak{g}$. All is not lost though, since if we add the map

$$f(u, \tilde{A}) : W_\perp^{1,p}(\mathfrak{A}, \mathfrak{g}) \times L^p(\mathfrak{A}, \mathfrak{g} \otimes T^*M) \to \mathfrak{g}$$

given by $f(u, \tilde{A}) = \int_{\mathfrak{A}} A_r = \int_{\mathfrak{A}} S^{-1}\frac{\partial}{\partial r}S + S^{-1}\tilde{A}_rS$, where again $S = \exp\ u$, then we see the map

$$\begin{aligned} (d^1Q(0,0), d^1f(0,0)) : W_\perp^{1,p}(\mathfrak{A}, \mathfrak{g}) &\to W^{-1,p}(\mathfrak{A}, \mathfrak{g}) \times \mathfrak{g} \\ u &\mapsto \left(\Delta u, \int_{\mathfrak{A}} \frac{\partial}{\partial r}u\right) \end{aligned}$$

is an isomorphism. We see this by noting that $u \mapsto (0,0) \implies u = \text{const}$ over the whole domain, not just on the boundary. Since u is L^2 perpendicular to the constants of $\mathfrak{g}$ over $\mathfrak{A}$, we must have $u = 0$. Therefore, we may solve for $d^*A = \int_{\mathfrak{A}} A_r = 0$ when $\tilde{A}$ is sufficiently small. Note that our solution is actually stronger, since $d^*A = 0$ implies that the integral $\int_{|x|=t} A_r$ is independent of t, and so the condition that $\int_{\mathfrak{A}} A_r = 0$ implies that $\int_{|x|=t} A_r = 0$ for every $t \in [1,2]$. □

Corollary 4.2.11. *(for $n > 2$). There exists a constant $\lambda(n)$ such that if d_A is a covariant derivative $d_A = d + A$ in $\mathfrak{A}$ with curvature $||F_A||_{L^\infty} < \gamma'$, $d^*A = 0$, $d^*_\psi A_\psi = 0$ and $\int_{|x|=r} A_r = 0$, then*

$$(\sqrt{\lambda(n)} - \kappa'||F||^2_{L^\infty}) \int_{\mathfrak{A}} |A|^2 \leq \int_{\mathfrak{A}} |F_A|^2,$$

where $\lambda(n)$ is the first eigenvalue of the Laplacian in n dimensions acting on A.

Proof. By definition, we can construct $\lambda(n)$ as

$$\lambda(n) = \inf_{f \neq 0} \frac{\int_{\mathfrak{A}} |df|^2}{\int_{\mathfrak{A}} |f|^2},$$

where $f \in W^{1,2}(T^*\mathfrak{A})$, $d^*f = 0$, $d^*_\psi f_\psi|_{S^{n-1}} = 0$. We claim then that $\lambda(n) > 0$. Clearly $\lambda(n) \geq 0$, and so assume that $\lambda(n) = 0$. Then there exists an f which satisfies the boundary conditions and $f \neq 0$ such that $df = 0$, i.e. f is a closed 1-form. Since $\mathfrak{A}$ is simply connected, every closed form is exact, which means that there exists a 0-form g such that $f = dg$. Note that since g is a 0-form $d^*g = 0$ by definition. On the boundary, we have $d^*_\psi d_\psi g = d^*_\psi f = 0$, so we must have $g = \text{constant}$ on the boundary. Since g is closed and co-closed, it is harmonic, i.e. $\Delta g = 0$ in $\mathfrak{A}$. Therefore g is a harmonic 0-form which is constant on the two boundaries. This implies that $g = c_1 + c_2 r^{n-2}$, but since $\int_{|x|=t} f_r = \int_{|x|=t} (n-2)c_2 t^{n-3} = 0$, we have $c_2 = 0$ for all $t \in [1,2]$. Therefore, g is constant on all of $\mathfrak{A}$, and so $f \equiv 0$ in $\mathfrak{A}$, which is a contradiction.

Once we have established that the first eigenvalue is strictly greater than 0, the corollary follows by calculations exactly the same as in Corollary 4.2.7. □

4.3 A Priori Estimates

In this section we prove some technical PDE lemmas which will be required in the proof of the removable singularities. The main argument is a version of the *Moser iteration* argument, which will allow us to gain stronger control of a function on an interior domain. We will assume that all covariant derivatives are smooth in some gauge. The difference in the cases where the Riemannian curvature is zero and non-zero is unimportant to our calculations, since it is only a lower order term, and so we will assume that M is flat. Then we have the following lemma:

Lemma 4.3.1. *If F is a Yang–Mills field, then*

$$|F|\Delta|F| \geq 2\langle F, [F, F]\rangle = 2\sum_{i,j,k}\langle F_{ij}, [F_{jk}, F_{ji}]\rangle$$
$$\Delta|F| \geq -4|F|^2$$

Proof. From the first Bianchi identity, equation (2.4.4), and the Yang–Mills equation, (2.9), we have that the Hodge Laplacian satisfies $\Delta_A F_A = 0$. From the Weitzenböck identity, equation 2.5, and since the underlying manifold is flat we see that these two operators differ by a curvature of the connection term when they act on two forms. Therefore, for any two form ψ,

$$(\nabla^2 - \Delta)\psi = [F, \psi] = \Big\{\sum_j [F_{ij}, \psi_{jk}] - [\psi_{ji}, F_{jk}]\Big\}$$

The crude Laplacian can be used to estimate a *scalar* Laplacian on the norm. We have the two expansions

$$d^*d|\psi|^2 = 2|\psi|d^*d|\psi| + 2|d|\psi||^2 = 2|\psi|\Delta|\psi| + 2|d|\psi||^2$$
$$d^*d|\psi|^2 = d^*d\langle\psi, \psi\rangle = 2\big(\langle\psi, \nabla^2\psi\rangle + \langle\nabla\psi, \nabla\psi\rangle\big),$$

Since ∇ is a metric connection and $|\psi|^2$ is scalar. Therefore, by equating the two expansions and applying Kato's inequality $|d|\psi|| \leq |\nabla\psi|^2$, we have

$$|\psi|\Delta|\psi| = \langle\psi, \nabla^2\psi\rangle + \langle\nabla\psi, \nabla, \psi\rangle - |d|\psi||^2 \geq \langle\psi, \nabla^2\psi\rangle$$

When $\psi = F$ we have that $(\nabla^2 - \Delta)F = [F, F]$. Since d_A is a Yang–Mills connection, we have $\Delta_A F_A = 0$, and so

$$|F|\Delta|F| \geq 2\langle F, [F, F]\rangle = 2\left\langle F_{ij}, \sum_{i,j,k}\Big\{[F_{ij}, F_{jk}] - [F_{ji}, F_{jk}]\Big\}\right\rangle$$

From the estimate

$$2\langle F, [F, F]\rangle \leq 2|F||[F, F]| \leq 4|F|^3,$$

it follows that

$$\Delta|F| \geq -4|F|^2.$$

□

We now regard $-4|F| = b$ as a fixed function and write the inequality as

$$\Delta f \geq -bf$$

in the weak sense for $f = |F|$. This which makes f a weak subsolution and we may now apply a Moser iteration technique to conclude a stronger control of f on the interior of the domain. Although long, the following theorem is crucial in the removable singularities argument, and so we thought it would be negligent to not include the fundamentals of the argument. For any domain $B_r(x_0) \subset B_R(x_0)$ we have the following theorem:

Theorem 4.3.2. *(Chapter 4, Theorem 1.1 of [23]) Suppose that $u \in H^1(B_R)$ is a weak subsolution in the following sense*

$$\int_{B_R} a_{ij} D_i u D_j \varphi + bu\varphi \leq 0 \tag{4.3}$$

for any $\varphi \in H_0^1(B_R)$ and $\varphi \geq 0$ in $B_R(x_0)$. Suppose $a_{ij} \in L^\infty(B_R)$ and $b \in L^q(B_R)$ for some $q > \frac{n}{2}$ satisfy the following assumptions:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \textit{ for any } x \in B_R,\ \xi \in \mathbb{R}^n,$$

and

$$||a_{ij}||_{L^\infty} + ||b||_{L^q} \leq \Lambda$$

for some positive constants λ and Λ.Then $u^+ \in L^\infty_{loc}(B_R)$. Moreover, there holds for any $\theta \in (0,1)$

$$\sup_{B_{\theta R}} u^+ \leq \frac{C}{(R-\theta R)^{\frac{n}{2}}} ||u^+||_{L^2(B_R)}$$

where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.

Remark 4.3.3. The D in the above theorem refers to the multi-index notation of PDE, not any sort of connection.

Proof of Theorem 4.3.2. The theorem must first be proven for the case $R = 1$ and $\theta = \frac{1}{2}$, and then the general case follows by a scaling argument. We follow the method as prescribed by [23].

For some $m, k > 0$, set $\bar{u} = u^+ + k$ and $\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k+m & \text{if } u \geq m \end{cases}$. Then we have $D\bar{u}_m = 0$ for $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Then choose the test function to be

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H^1(B_1)$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. We then calculate

$$D\varphi = \beta\eta^2\bar{u}_m^{\beta-1}\bar{u}D\bar{u}_m + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}).$$

Since for $u \geq m$ we have $D\bar{u}_m = 0$, and for $u < m$ we have $\bar{u}_m = \bar{u}$, we may rewrite this as

$$D\varphi = \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}).$$

Next, we substitute this into (4.3) and expand. Note that $u^+ \leq \bar{u}$, $\bar{u}_m^\beta \bar{u} - k^{\beta+1} \leq \bar{u}_m^\beta \bar{u}$ and that by the ellipticity condition we have

$$a_{ij}(x) D_j \bar{u}_m D_i \bar{u}_m \geq \lambda |D\bar{u}_m|^2.$$

Therefore

$$\begin{aligned}
\int_{B_R} a_{ij} D_i u D_j \varphi &= \int_{B_R} a_{ij} D_i \bar{u} (\beta D_j \bar{u}_m + D_j \bar{u}) \eta^2 \bar{u}_m^\beta \\
&\qquad + 2 \int_{B_R} a_{ij} D_i \bar{u} D_j \eta (\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \eta \\
&\geq \lambda \beta \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \lambda \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \\
&\qquad - 2\Lambda \int_{B_R} |D\bar{u}||D\eta| \bar{u}_m^\beta \bar{u} \eta.
\end{aligned}$$

Then, note that by the Hölder inequality and Young's inequality with ε, where $\varepsilon = \frac{\lambda}{2\Lambda}$

$$\begin{aligned}
2\Lambda \int_{B_R} |D\bar{u}| \bar{u}_m^{\frac{\beta}{2}} \eta |D\eta| \bar{u}_m^{\frac{\beta}{2}} \bar{u} &\leq 2\Lambda \Big(\int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \Big)^{\frac{1}{2}} \Big(\int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 \Big)^{\frac{1}{2}} \\
&\leq \frac{\lambda}{2} \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 + \frac{2\Lambda^2}{\lambda} \int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2,
\end{aligned}$$

we have

$$\int_{B_R} a_{ij} D_i u D_j \varphi \geq \lambda \beta \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{\lambda}{2} \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - \frac{2\Lambda}{\lambda} \int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2.$$

Therefore, from this calculation and from (4.3) for $C = C(\lambda, \Lambda)$, we have

$$\begin{aligned}
\beta \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 &\leq C \Big\{ \int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int_{B_R} a_{ij} D_i u D_j \varphi \Big\} \\
&\leq C \Big\{ \int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int_{B_R} |b| \bar{u} \varphi \Big\} \\
&\leq C \Big\{ \int_{B_R} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int_{B_R} |b| \bar{u}^2 \eta^2 \bar{u}_m^\beta \Big\}
\end{aligned}$$

where the last inequality follows since $\varphi = \eta^2 (\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \leq \eta^2 \bar{u}_m^\beta \bar{u}$, since $\bar{u} > k$. Now, set $w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}$. Now, by Young's inequality with $\varepsilon = 2\beta$, we have

$$\begin{aligned}
|Dw|^2 &= \Big| \frac{\beta}{2} \bar{u}_m^{\frac{\beta}{2}-1} \bar{u} D\bar{u}_m + \bar{u}_m^{\frac{\beta}{2}} D\bar{u} \Big|^2 \\
&\leq \frac{\beta^2}{4} \bar{u}_m^\beta |D\bar{u}_m|^2 + \beta \bar{u}_m^\beta |D\bar{u}_m||D\bar{u}| + \bar{u}_m^\beta |D\bar{u}|^2 \\
&\leq \frac{\beta^2}{4} \bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{\beta}{4} \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta \beta |D\bar{u}|^2 + \bar{u}_m^\beta |D\bar{u}|^2 \\
&\leq (\beta + 1) \big\{ \beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2 \big\}.
\end{aligned}$$

From this we have

$$\int_{B_R} |Dw|^2\eta^2 \leq C\Big\{(1+\beta)\int_{B_R} \bar{u}_m^\beta \eta^2 |D\bar{u}_m|^2 + (1+\beta)\int_{B_R} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2\Big\}$$
$$\leq C\Big\{(1+\beta)\int_{B_R} w^2|D\eta|^2 + (1+\beta)\int_{B_R} |b|w^2\eta^2\Big\},$$

and so

$$\int_{B_R} |D(w\eta)|^2 \leq C\Big\{(1+\beta)\int_{B_R} w^2|D\eta|^2 + (1+\beta)\int_{B_R} |b|w^2\eta^2\Big\}.$$

By assumption, we have $||b||_{L^q} \leq \Lambda$, and a simple application of the Hölder inequality yields

$$\int_{B_R} |b|w^2\eta^2 \leq \Big(\int_{B_R} |b|^q\Big)^{\frac{1}{q}}\Big(\int_{B_R} (\eta w)^{\frac{2q}{q-1}}\Big)^{1-\frac{1}{q}} \leq \Lambda\Big(\int_{B_R} (\eta w)^{\frac{2q}{q-1}}\Big)^{1-\frac{1}{q}}.$$

By the interpolation inequality, $||u||_{L^q} \leq \varepsilon||u||_{L^r} + \varepsilon^{-\mu}||u||_{L^p}$, where $r \geq q \geq p$, we have

$$||\eta w||_{L^{\frac{2q}{q-1}}} \leq \varepsilon||\eta w||_{L^{2^*}} + \varepsilon^{-\mu}||\eta w||_{L^p},$$

where $\mu = \frac{\left(\frac{1}{p}-\frac{1}{q}\right)}{\left(\frac{1}{q}-\frac{1}{r}\right)}$ and $\frac{1}{p}+\frac{1}{q} = 1$. Therefore, we have $p = \frac{q+1}{2q}$ and $\mu = \frac{n}{2q-n}$. Since $p < 2$ and by the Gagliardo-Nirenberg-Sobolev inequality (see, for example, Chapter 9.3 of [10]), we have

$$||\eta w||_{L^{\frac{2q}{q-1}}} \leq \varepsilon||D(\eta w)||_{L^2} + C(n,q)\varepsilon^{\frac{-n}{2q-n}}||\eta w||_{L^2}$$

for small ε. Therefore, we may choose ε such that

$$\int_{B_R} |D(w\eta)|^2 \leq C\Big\{(1+\beta)\int_{B_R} w^2|D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}}\int_{B_R} w^2\eta^2\Big\},$$

and in particular

$$\int_{B_R} |D(w\eta)|^2 \leq C(1+\beta)^\alpha \int_{B_R} (|D\eta|^2+\eta^2)w^2,$$

where $\alpha = \alpha(n,q) > 0$. Now, by the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\Big(\int_{B_R} |w\eta|^{2\chi}\Big)^{\frac{1}{\chi}} \leq \int_{B_R} |D(w\eta)|^2 \leq C(1+\beta)^\alpha \int_{B_R} (|D\eta|^2+\eta^2)w^2,$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$. We may then choose the cutoff function as follows: For any $0 < r < R$, set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_R \qquad \text{and} \qquad |D\eta| \leq \frac{2}{R-r}.$$

We then obtain

$$\left(\int_{B_r} |w|^{2\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2.$$

Since we defined $w = \bar{u}_m^{\frac{\beta}{2}}\bar{u}$, we then also have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}_m^{\beta}\bar{u}^2.$$

Note that $\bar{u}_m \leq \bar{u}$, and so if we set $\gamma = \beta + 1$, we obtain

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^{\gamma},$$

provided the integral on the right hand side is bounded. By letting $m \to \infty$, we conclude that

$$||\bar{u}||_{L^{\gamma\chi}(B_r)} \leq \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{\frac{1}{\gamma}}||\bar{u}||_{L^{\gamma}(B_R)},$$

provided $||\bar{u}||_{L^{\gamma}(B_R)} < \infty$, where $C = C(n, q, \lambda, \Lambda) > 0$ and is independent of γ.

This is the crucial step of the proof, in that we have obtained higher integrability of a function on for the price of having to restrict to a smaller ball. Such a process suggests an iterative argument, which we now demonstrate. We begin the iteration with $\gamma = 2\text{m}$ and iterate as

$$\gamma_i = 2\chi^i \qquad \text{and} \qquad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$$

for $i = 0, 1, 2, ...$ By noting that $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = \frac{1}{2^{i+1}}$, we have for $i = 1, 2, 3, ...,$

$$||\bar{u}||_{L^{\gamma_i}(B_{r_i})} \leq C(n, q, \lambda, \Lambda)^{\frac{1}{\chi^i}}||\bar{u}||_{L^{\gamma_{i-1}}(B_{r_{i-1}})}.$$

By continuing this iteration, we obtain

$$||\bar{u}||_{L^{\gamma_i}(B_{r_i})} \leq C^{\sum\frac{1}{\chi^i}}||\bar{u}||_{L^{\gamma_{i-1}}(B_{r_{i-1}})},$$

and in particular

$$\left(\int_{B_{\frac{1}{2}}} \bar{u}^{2\chi^i}\right)^{\frac{1}{2\chi^i}} \leq C\left(\int_{B_1} \bar{u}^2\right)^{\frac{1}{2}}.$$

By taking $i \to \infty$ and recalling the definition of the L^∞ norm, we find that

$$\sup_{B_{\frac{1}{2}}} \bar{u} \leq C||\bar{u}||_{L^2(B_1)}.$$

We now let $k \to 0$, and obtain

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C||u^+||_{L^2(B_1)}.$$

This completes the proof for the case of $r = \frac{1}{2}$, $R = 1$. The general result follows by a dilation argument, although we refer to pp 79 of [23] for the completion of this proof. □

Remark 4.3.4. Theorem 4.3.2 actually applies in greater generality than we have shown, although to show this in all of its generality would be superfluous to our needs. For the full theorem, we refer to Theorem 1.1 of Chapter 4 of [23], although the theorem is well known.

Remark 4.3.5. The above theorem will be crucial in proving boundedness of the curvature when the L^q norm is finite. Since we will be assuming that the singularity occurs at the origin, the trade-off to L^∞ regularity on a smaller ball is justified, since the ball can be made arbitrarily small and still contain the singularity.

The next two lemmas are by-products of the above calculation, although we state and prove them explicitly for clarity.

Lemma 4.3.6. *Let $U \subset \mathbb{R}^n$, $f \in W^{1,2}_{loc}(U) \cap L^\infty_{loc}(U)$, $f \geq 0$, $1/2 < p < \infty$, $\nu = \frac{2n}{n-2}$ and $u \in C^\infty_0(U)$. Then if*

$$-\Delta f \leq bf,$$

$$\int_U |d(uf^p)|^2 \leq \int_U \left[\frac{p|p-1|}{2p-1}|\Delta u^2| + (du)^2\right] f^{2p} + \frac{p^2}{2p-1}\Big(\int_U b^{2/n}\Big)^{n/2}\Big(\int_U (uf^p)^\nu\Big)^{2/\nu}.$$

Proof. We may replace f with $f + \varepsilon$, prove the estimate for $f + \varepsilon$ and then let $\varepsilon \to 0$. Take $u^2 f^{2p-1}$ as a test function. Then

$$\int_U d(u^2 f^{2p-1}) \cdot df = -\int_U u^2 f^{2p-1}\Delta f \leq \int_U bu^2 f^{2p}$$

Notice that

$$\begin{aligned} d(u^2 f^{2p-1}) \cdot df &= f^{2p-1} du^2 \cdot df + u^2(2p-1) f^{2p-2} df \cdot df \\ &= \frac{1}{2p} du^2 \cdot df^{2p} + (2p-1)u^2 f^{2p-2}|df|^2, \end{aligned}$$

and

$$\begin{aligned} d(uf^p) &= f^p du + pf^{p-1} u df \\ &\implies |d(uf^{2p})|^2 = f^{2p}|du|^2 + p^2 f^{2p-2}u^2|df|^2 + \frac{1}{2}du^2 \cdot f^{2p} \\ &\implies |df|^2 f^{2p-2}u^2 = \frac{d(uf^p)|^2}{p^2} - \frac{f^{2p}|du|^2}{p^2} - \frac{du^2 \cdot df^{2p}}{2p^2} \end{aligned}$$

Upon substitution, we find that

$$d(u^2 f^{2p-1})\cdot df = \frac{2p-1}{p^2}|d(uf^p)|^2 - \frac{p-1}{2p^2}(du^2\cdot df^{2p}) - \frac{2p-1}{p^2}|du|^2 f^{2p}$$

The right hand side can be estimated by Hölder's inequality, giving

$$\int_U b(uf^p)^2 \leq \Big(\int_U b^{n/2}\Big)^{2/n}\Big(\int_U (uf^p)^\nu\Big)^{2/\nu}.$$

This gives us

$$\begin{aligned}\frac{2p-1}{p^2}\int_U |d(uf^p)|^2 \leq & \frac{|p-1|}{p}\int_U du^2\cdot d(f^{2p}) \\ & + \frac{2p-1}{p^2}\int_U |du|^2 f^{2p} + \Big(\int_U b^{n/2}\Big)^{2/n}\Big(\int_U (u^2 f^p)^\nu\Big)^{2/\nu}\end{aligned}$$

If we integrate the first term on the right by parts and multiply the entire equation by $p^2/(2p-1)$ we get the inequality as stated in the lemma. □

Lemma 4.3.7. *Assume the conditions of Lemma 4.3.6, and suppose for $q \leq 1$ there exists a constant c_n such that if $B(x_0, a_0) \subset U$,*

$$c_n - \Big(\int_U |b|^{n/2}\Big)^{2/n}\frac{q^2}{2q-1} > \gamma > 0.$$

Then for all $B(x, 2a) \subset U$, we have $uf^q \in W^{1,2}(B(x,a))$ with

$$a^{-n+2}\int_{B(x,a)} |df^q|^2 \leq c_\gamma a^{-n}\int_{B(x,2a)} |f|^2$$

$$\Big(a^{-n}\int_{B(x,a)} |f|^{q\nu}\Big)^{2/q\nu} \leq c'_\gamma a^{-n}\int_{B(x,2a)} |f|^2.$$

Furthermore, c_γ and c'_γ depend only on γ, q and n. u is a smooth cutoff function which is one on $B(x,a)$ and 0 on $B(x,2a)$.

Proof. Lemma 4.3.6 applies with $U = B(x, 2a)$ and $1 \leq p \leq q$. Since the integral is dilation invariant, we may asume WLOG that $a = 1$. For ease of notation, let $\mathscr{K}(u,p) = \max[\frac{p|p-1|}{2p-1}|\Delta u^2| + (du)^2]$. By applying the Gagliardo-Nirenberg-Sobolev inequality with $\nu = \frac{2n}{n-2}$,

$$\begin{aligned} c_n\Big(\int_U |uf^p|^\nu\Big)^{2/\nu} \leq \int_U |d(uf^p)|^2 \leq \mathscr{K}(u,p)\int_U f^{2p} \\ + \frac{p^2}{2p-1}\Big(\int_U |b|^{n/2}\Big)^{2/n}\Big(\int_U |uf^p|^\nu\Big)^{2/\nu}\end{aligned}$$

For $q \geq p \geq 1$, $\frac{p^2}{2p-1}$ is monotonically increasing, and so $\frac{p^2}{2p-1} \leq \frac{q^2}{2q-1}$. This, with the hypothesis of the lemma, gives

$$\gamma\Big(\int_U |uf^p|^\nu\Big)^{2/\nu} \leq \mathscr{K}(u,p)\int_U |f|^{2p}$$

and

$$\begin{aligned}\frac{\gamma}{c_n}\int_U |d(uf^p)|^2 &\leq \frac{\gamma}{c_n}\bigg(\mathscr{K}(u,p)\int_U |f|^{2p} \\ &\quad + \frac{p^2}{2p-1}\Big(\int_U |b|^{n/2}\Big)^{2/n}\Big(\int_U |uf^p|^\nu\Big)^{2/\nu}\bigg) \\ &\leq \frac{\gamma}{c_n}\bigg(\mathscr{K}(u,p)\int_U |f|^{2p} + (c_n-\gamma)\Big(\int_U |uf^p|^\nu\Big)^{2/\nu}\bigg) \\ &\leq \frac{\gamma}{c_n}\mathscr{K}(u,p)\int_U |f|^{2p} + \frac{(c_n-\gamma)}{c_n}\mathscr{K}(u,p)\int_U |f|^{2p} \\ &\leq \mathscr{K}(u,p)\int_U |f|^{2p} \leq \mathscr{K}(u,q)\int_U |f|^{2p}.\end{aligned}$$

Since u is a smooth cutoff function, this gives us a bound on the $L^{\nu p}$ norm on domains interior to U by the L^{2p} norm on the domain. Starting with $p_0 = 1$, we iterate the construction and can achieve $q = p_i$ in a finite number of steps, thus obtaining the result. □

Theorem 4.3.8. *There exists a constant c_n' such that if F is a Yang–Mills field in $B(x_0, 2a_0)$ and $\int_{B(x_0,2a_0)} |F|^{n/2} < c_n'$, then $|F(x)|$ is uniformly bounded in the interor of $B(x_0, 2a_0)$ and*

$$|F(x)|^2 \leq a^{-n}\mathscr{K}_n \int_{B(x,a)} |F|^2.$$

Proof. Let $b = 4|F|$ and $|F| = f$. Then, by letting $c_n' = \left(\frac{c_n}{4n}\right)^{\frac{n}{2}}$, $\gamma = \frac{c_n}{3}$, $q = n$, for $n \geq 2$, we have that $\frac{2}{3} > \frac{n}{2n-1}$, and so we may apply Lemma 4.3.7. This then gives us a bound on $||F||_{L^{n\nu}}$, which gives us a bound on $||F||_{L^n}$. We may then apply Theorem 4.3.2, which gives us the desired result. Note that the constants c_n and $\mathscr{K}_n$ are not affected by the size of the ball, since the integral is invariant under dilation. □

Theorem 4.3.9. *Let F be a smooth Yang–Mills field in a punctured ball $U = B(x_0, a) - \{x_0\}$ such that $\int_U |F|^q < \infty$ for $q > \max(\frac{n}{n-1}, \frac{n}{2})$. Then $|F_A|$ is uniformly bounded in the interior of $B(x_0, a_0)$.*

Before giving the proof we first explain the idea behind it. In Lemmas 4.3.6 and 4.3.7 we were able to find bounds of $||uf^p||_{W^{1,2}}$, where u is a suitable test function so that $||uf^p||_{W^{1,2}}$ does not blow up. We would like to use this construction, but keep control of exactly how much u 'helps' $||uf^p||_{W^{1,2}}$ not blow up. In doing so we test exactly how close the test function can come to being arbitrary whilst keeping control of $||uf^p||_{W^{1,2}}$. With this approach we find that the hypothesis of the Theorem are exactly the conditions needed to enforce that $||f^p||_{W^{1,2}} < \infty$. This then allows us to use Theorem 4.3.2 to conclude an L^∞ bound.

Proof of Theorem 4.3.9. Let $b = 4|F|$, $f = |F|$ and $U = B(x_0, a) - \{x_0\}$ and apply Lemmas 4.3.6 and 4.3.7. Here, let $u = v + v'$, where v is a cut-off function which is zero at x_0, $v' \in C_0^\infty(B(x_0, a))$. Firstly, we fix v' in order to deal exclusively with the badly-behaved part of the function. Let $v(x - x_0) = \varphi(\frac{x}{\varepsilon})$, where ε has support in the unit ball. By doing this, we ensure that the support of v vanishes as $\varepsilon \to 0$. In Lemma 4.3.6 we would like to control the right hand side to not blow up as we take $\varepsilon \to 0$. This will mean that our test function does not need to be zero at the singularity, and we will then have $f^p \in W^{1,2}(B(x,a))$. On the right hand side of

$$\int_U |d(uf^p)|^2 \leq \int_U \left[\frac{p|p-1|}{2p-1}|\Delta u^2| + (du)^2\right] f^{2p} + \frac{p^2}{2p-1}\Big(\int_U b^{2/n}\Big)^{n/2}\Big(\int_U (uf^p)^\nu\Big)^{2/\nu},$$

we have that

$$\Big(\int_U b^{2/n}\Big)^{n/2}\Big(\int_U (uf^p)^\nu\Big)^{2/\nu} < \infty$$

for arbitrary u by hypothesis, and so we must consider the potential blow-up of $\mathscr{K}(u,p)\displaystyle\int_U f^{2p}$. Since v' is fixed we have that $\mathscr{K}(u,p) \sim \mathscr{K}(v,p) \sim \varepsilon^{-2}K(\varphi,p)$. We then have that

$$\begin{aligned}\mathscr{K}(u,p)\int_U f^{2p} &\leq \varepsilon^{-2}\mathscr{K}(\varphi,p)\int_U f^{2p} \\ &\leq \varepsilon^{-2}\mathscr{K}(\varphi,p)\int_{|x-x_0|\leq\varepsilon} f^{2p} \\ &\leq \mathscr{K}(\varphi,p)\varepsilon^{n(1-\frac{2p}{q})-2}\int_U (f^q)^{\frac{2p}{q}},\end{aligned}$$

where the indices of ε arise from dilation back to U and the change of index of f. We see then, that if $n(1-\frac{2p}{q})-2>0$, then the contribution from $v' \to 0$ as $\varepsilon \to 0$ and the test function chosen is arbitrary. So that $\mathscr{K}(u,p)$ doesn't blow up due to p we require that $p > \frac{1}{2}$, and this is clearly only possible if and only if $q > \frac{n}{n-2}$. By Lemmas 4.3.6 and 4.3.7 with $U = B(x_0,a)$ we find that $f^p \in W^{1,2}(x_0,a)$. Since we have that $q > \frac{n}{2}$ by hypothesis, we may now apply Theorem 4.3.2 to find a bound on $|F_A|$ in $B(x_0,a)$. □

4.4 Removability of Singularities

In this section we utilise all the machinary which has been built up in the previous two sections to prove the following removable singularities theorem.

Theorem 4.4.1. *Let d_A be a Yang–Mills connection in a bundle E over $B^4-\{0\}$. If the L^2 norm of the curvature F_A of d_A is finite, $\displaystyle\int_{B^4}|F_A|^2 < \infty$, then there exists a gauge in which the bundle E extends to a smooth bundle $\bar{E}$ over B^4 and the connection d_A extends to a smooth Yang–Mills connection $d_{\bar{A}}$ in $\bar{E}$.*

Proof. As with the proof of the existence of Coulomb gauges, we will approach the proof of removable singularities in steps. The first step is to use the gauge construction of Section 4.2 to construct a gauge over $B^4-\{0\}$. We do this by splitting the unit ball into a dyadic domain and constructing a Coulomb gauge on each of these annuli such that they satisfy a continuity condition on the boundary. We call this a *broken* Coulomb gauge. We then use the Yang–Mills equation on this broken Coulomb gauge to show that the curvature is uniformly bounded over the origin. We may then apply Theorem 4.2.8 to complete the proof. The formal construction and definition of the broken Coulomb gauge is as follows: Let

$$\mathfrak{A}_\ell = \{x : 2^{-\ell-1} \le |x| \le 2^{-\ell}\} \qquad \text{for } \ell = \{0, 1, 2, ...\}$$
$$S_\ell = \{x : |x| = 2^{-\ell}\} \qquad \text{for } \ell = \{0, 1, 2, ...\}$$

Definition 4.4.2. A broken Coulomb gauge for a connection d_A in a bundle E over $B^n - \{0\}$ is a gauge related to the original gauge in which $d_A = d + A$ and $A|_{\mathfrak{A}_\ell} = A(\ell)$ have the following properties for all $\ell \ge 0$

(i) $d^* A(\ell) = 0$ in $\mathfrak{A}_\ell$

(ii) $A_\psi(\ell)|_{S_\ell} = A_\psi(\ell - 1)|_{S_\ell}$

(iii) $d^*_\psi A_\psi(\ell) = 0$ on S_ℓ and $S_{\ell+1}$

(iv) $\displaystyle\int_{S_\ell} A_r(\ell) = \int_{S_{\ell+1}} A_r(\ell) = 0.$

Note that (i) means that the gauge is Coulomb gauge in $\mathfrak{A}_\ell$. Condition (ii) implies that the induced connection on the pull-back bundle $E|_{S_\ell}$ is the same from the gauges given in $\mathfrak{A}_\ell$ and $\mathfrak{A}_{\ell-1}$, although this is actually insured by the condition that the gauge be continuous. Condition (iii) ensures that the gauge is still Coulomb independently on the sphere, and condition (iv) allows us to apply Theorem 4.2.10 and Corollary 4.2.11.

Claim 4.4.3. *If the hypotheses of 4.4.1 hold, given any $\varepsilon > 0$, we may assume* $\displaystyle\int_{B(0,2)} |F|^2 \le \varepsilon^2$

Proof. If $\displaystyle\int_{B^4} |F|^2 < \infty$, then $\displaystyle\lim_{r\to 0} \int_{|x|\le r} |F|^2 = 0$. Assume then, that $\displaystyle\int_{|x|\le\rho} |F| \le \varepsilon^2$. We then change co-ordinates by $y = \frac{2x}{\rho}$. Then $F(x)$ pulls back to a Yang–Mills field $\tilde{F}(y)$ on $\{y : 0 < |y| \le 2\}$ and

$$\int_{B(0,2)} |\tilde{F}(y)|^2 dy = \int_{|x|\le\rho} |F(x)|^2 dx \le \varepsilon^2.$$

By the dilation property of the Lebesgue integral. Note, however, that the uniformity of the estimates is lost in passing from $\tilde{F}$ back to F. □

Claim 4.4.4. *Under the hypotheses of 4.4.1, if* $\displaystyle\int_{B(0,2)} |F|^2 \le C'$, *then*

$$|F(x)|^2 \le |x|^{-4} k \int_{B(0,2|x|)} |F|^2$$

for $|x| \le 1$. Here $C' = C'_4$ and $k = k_4$ are the constants from Theorem 4.3.8 with $n = 4$.

Proof. If $|x| \le 1$, then $B(x, |x|) \subset B(0, 2)$ and

$$\int_{B(x,|x|)} |F|^2 \le \int_{B(0,2)} |F|^2 \le C'_4$$

We can then apply Theorem 4.3.8 to achieve the result. □

The following is true in all dimensions.

Claim 4.4.5. *There exists $\gamma'(=\gamma_n) > 0$ such that if d_A is a smooth connection in $B^n - \{0\}$, and the growth on the curvature satisfies $|F(x)||x|^2 \leq \gamma \leq \gamma'$, then there exists a broken Coulomb gauge in $B^n - \{0\}$ satisfying*

(v) $|A(\ell)(x)| \leq \kappa' ||F_{A(\ell)}||_{L^\infty} 2^{-(\ell+1)} \leq \kappa' \gamma 2^{\ell+1}$

(vi) $(\lambda(n) - k^2\omega^2) \int_{\mathfrak{A}(\ell)} |A(\ell)|^2 \leq 2^{-2(\ell+1)} \int_{\mathfrak{A}(\ell)} |F_{A(\ell)}|^2$

Proof. By dilating each annulus $\mathfrak{A}_\ell$ by $y = x2^{\ell+1}$, we bring $\mathfrak{A}_\ell$ into $\mathfrak{A}$ of Theorem 4.2.10, on which the curvature is denoted $\tilde{F}(y)$. In local co-ordinates, $F(x) = F_{ij}(x)dx^i \wedge dx^j$ is a two form on $\mathfrak{A}_\ell$, and $f = \frac{y}{2^{\ell+1}}$ is a smooth mapping from $\mathfrak{A}$ to $\mathfrak{A}_\ell$, so the pullback $f^*F(x)$ is a two form on $\mathfrak{A}$, given by $\tilde{F}(y) = 2^{-2(\ell+1)} F_{ij}(f(y))dy^i \wedge dy^j$. From this, we see that $|\tilde{F}(y)| = 2^{-2(\ell+1)}|F(x)|$. Therefore, $|\tilde{F}(y)||y|^2 = |F(x)||x|^2 \leq \gamma$, and so $||\tilde{F}||_{L^\infty} \leq \gamma$. We may then apply Theorem 4.2.10 to $d_{A(\ell)}$ to extract a gauge on which $(i) - (iv)$ are true, and then dilate the annulus back to the original one. It is not immediately obvious that it is necessary that the construction of gauges on the annuli must agree on boundary spheres. Recall from Theorem 4.2.6 that the choice of gauge on a sphere where $d^*_\psi A_\psi = 0$ is unique up to multiplication by a constant element of G. Therefore the gauge constructed on $\mathfrak{A}_\ell$ and $\mathfrak{A}_{\ell-1}$ differ by a constant element $g_\ell \in G$ on their mutual boundary S_ℓ. Therefore, rotate the gauge on $\mathfrak{A}_\ell$ by $h_\ell = g_\ell g_{\ell-1} \ldots g_1$. By Theorem 4.2.10 we have $||A(y)||_{L^\infty} \leq \kappa' ||F(y)||_{L^\infty} \leq \kappa'\gamma$, and so after dilating back to the original anulus, we have condition (v). Condition (vi) follows by applying Corollary 4.2.11 and then dilating. □

We now restrict our attention to 4 dimensions again.

Claim 4.4.6. *Let $n = 4$. Then there exists $\varepsilon > 0$ such that if d_A is a Yang–Mills connection in $B(2,0) - \{0\}$ and $\displaystyle\int_{B(2,0)} |F|^2 \leq \varepsilon^2$ then*

$$\left(1 - \omega\Big(\int_{|x|\leq 2r} |F_A|^2\Big)^{1/2}\right)\left(\int_{|x|\leq r} |F_A|^2\right) \leq \frac{1}{4} r \int_{|x|=r} |F_A|^2.$$

Proof. From Claim 4.4.4, if we choose $\varepsilon^2 \leq C'$, then we may apply Claim 4.4.4 to get

$$|F(x)|^2 \leq k|x|^{-4} \int_{\mathfrak{A}} |F|^2.$$

Since $\displaystyle\int_{\mathfrak{A}} |F|^2 \leq \varepsilon^2$, we get $|F(x)||x|^2 \leq \sqrt{k\varepsilon^2}$. Therefore, if $\sqrt{k\varepsilon^2} \leq \min(\gamma_0, \gamma_1, \gamma')$, we may then apply Claim 4.4.5 to yield the existence of a broken Coulomb gauge on $B(2,0) - \{0\}$. By integration by parts, we may then estimate

$\displaystyle\int_{\mathfrak{A}} |F|^2$ in the Coulomb gauge. We will estimate the curvature on each annulus $\mathfrak{A}_\ell$.

$$\begin{aligned}\int_{\mathfrak{A}_\ell} |F_{A(\ell)}|^2 &= \int_{\mathfrak{A}_\ell} \langle dA(\ell) + A(\ell)\wedge A(\ell), F_{A(\ell)}\rangle \\ &= \int_{\mathfrak{A}_\ell} \langle d_A A(\ell) - A(\ell)\wedge A(\ell), F_{A(\ell)}\rangle,\end{aligned}$$

Going from the first to the second equality is not at all obvious. Note, however, that since $A(\ell)$ is a one-form, we have $d_A(A\cdot s) = (d_A A)\cdot s - A\cdot(d_A s)$, and so $(d_A A)\cdot s = d_A(A\cdot s) + A\cdot(d_A s)$, which is the same as

$$\begin{aligned}(d_A A)\cdot s &= (d+A)(As) + A((d+A)s) \\ &= (dA)s - A\wedge ds + (A\wedge A)s + A\wedge ds + (A\wedge A)s \\ &= (dA)s + 2(A\wedge A)s,\end{aligned}$$

and so we see that $d_A A - A\wedge A = F_A$, which gives the second equality. By the cyclic property of the trace norm introduced in Section 2.7 and integration by parts, we have

$$\begin{aligned}\int_{\mathfrak{A}_\ell} |F_{A(\ell)}|^2 = &\int_{\mathfrak{A}_\ell} \langle A(\ell), -(d_A^* F_{A(\ell)} + A(\ell)\wedge F_{A(\ell)})\rangle \\ &+ \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle.\end{aligned}$$

Where the boundary terms with A_r in them go to zero by Hölder's inequality and the fact that $\displaystyle\int_{S_\ell} A_r = 0$, and the $\langle A_\psi, F_{\psi\psi}\rangle$ term doesn't appear since it's not a normal term. We then sum this over $\ell \geq 0$, and we see that the boundary terms are telescoping and this leaves only S_0, since $A_\psi(\ell) = A_\psi(\ell-1)$ on S_ℓ and the curvature F_A is continuous across S_ℓ. Since d_A is a Yang–Mills connection, we have $d^*_{A(\ell)}F_{A(\ell)} = 0$, and so

$$\begin{aligned}\int_{\mathfrak{A}_\ell} \langle d_{A(\ell)}A(\ell) - A(\ell)\wedge A(\ell), F_{A(\ell)}\rangle &= \int_{\mathfrak{A}_\ell} \langle A(\ell), -A(\ell)\wedge F_{A(\ell)}\rangle \\ &\quad + \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle \\ &= -\int_{\mathfrak{A}_\ell} \langle A(\ell)\wedge A(\ell), F_{A(\ell)}\rangle \\ &\quad + \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle \\ \int_{\mathfrak{A}_\ell} \langle d_{A(\ell)}A(\ell), F_{A(\ell)}\rangle &= \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle \\ \int_{\mathfrak{A}_\ell} \langle F_{A(\ell)} + A(\ell)\wedge A(\ell), F_{A(\ell)}\rangle &= \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle.\end{aligned}$$

Moreover, the boundary terms become negligible as $\ell\to\infty$; $\displaystyle\lim_{\ell\to\infty}\int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle = 0$. This is because we have that $|F_A| \sim \frac{1}{|x|^2}$ by Claim 4.4.4, $|A_\psi(\ell)|$ is bounded and that $\mathrm{vol}(S_\ell) \sim |x|^3$ as $|x|\to 0$. Therefore the $|x|^3$ terms dominates the $\frac{1}{|x|^2}$ term and we have that the integral goes to zero as $|x|$ tends to zero. Therefore, summing over ℓ, we have

$$\sum_{\ell=0}^{\infty}\int_{\mathfrak{A}_\ell}\langle F_{A(\ell)}+A(\ell)\wedge A(\ell),F(\ell)\rangle=\int_{S_0}\langle A_\psi(0),F_{r\psi}(0)\rangle$$
$$\leq\left(\int_{S_0}|A_\psi|^2\right)^{\frac{1}{2}}\left(\int_{S_0}|F_{r\psi}|^2\right)^{\frac{1}{2}}. \tag{4.4}$$

By construction, the restriction of the connection to $\eta|_{S_0}$ is given by $d_{A_\psi}=d_\psi+A_\psi$ with $d_\psi^*A_\psi=0$. Applying Corollary 4.2.7 yields

$$(2-K||F||_{L^\infty})\int_{S_0}|A_\psi|^2\leq\int_{S_0}|F_{\psi\psi}|^2.$$

Note that there is no $F_{r\psi}$ component because we are restricting to the sphere, so $F_A=F_{\psi\psi}$. Analogously, by Corollary 4.2.11 we have

$$\left|\int_{\mathfrak{A}_\ell}\langle F_{A(\ell)},A(\ell)\wedge A(\ell)\rangle\right|\leq||F_{A(\ell)}||_{L^\infty}\int_{\mathfrak{A}_\ell}|A(\ell)|^2$$
$$\leq 2^{-2(\ell+1)}||F(\ell)||_{L^\infty}(\lambda_4-\kappa' 2^{-4(\ell+1)}||F_{A(\ell)}||^2_{L^\infty})^{-1}\int_{\mathfrak{A}_\ell}|F_{A(\ell)}|^2,$$

where the factor of $2^{-2(\ell+1)}$ again arises from the dilation from $\mathfrak{A}_\ell$ to the standard annulus. By Claim 4.4.4, we have

$$2^{-2(\ell+1)}||F_{A(\ell)}||_{L^\infty}\leq\sqrt{k}\left(\int_{|x|\leq 2^{-\ell}}|F_{A(\ell)}|^2\right)^{\frac{1}{2}}\leq\sqrt{k}\left(\int_{|x|\leq 2}|F_A|^2\right)^{\frac{1}{2}}\leq\varepsilon\sqrt{k},$$

and if we assume that $\kappa' k\varepsilon^2\leq\frac{\lambda_4}{2}$, then we get the simplified estimate

$$\left|\int_{\mathfrak{A}_\ell}\langle F_{A(\ell)},A(\ell)\wedge A(\ell)\rangle\right|\leq\frac{2\sqrt{k}}{\lambda_4}\left(\int_{|x|\leq 2}|F_A|^2\right)^{\frac{1}{2}}\int_{\mathfrak{A}_\ell}|F_{A(\ell)}|^2.$$

Note that although $B^4=\bigcup_{\ell=0}^{\infty}\mathfrak{A}_\ell\cup\{0\}$, we have that

$$\int_{B^4}|F_A|^2=\int_{\bigcup_{\ell=0}^{\infty}\mathfrak{A}_\ell}|F_A|^2,$$

since $\{0\}$ has (Lebesgue) measure zero. Putting all these estimates together into (4.4), we have

$$\int_{B=\bigcup\mathfrak{A}_\ell}|F_A|^2\leq\frac{2\sqrt{k}}{\lambda_4}\left(\int_{|x|\leq 2}|F_A|^2\right)^{\frac{1}{2}}\int_{\sum\mathfrak{A}_\ell}|F_A|^2+$$
$$\left(2-\sqrt{k}K\left(\int_{|x|\leq 2}|F_A|^2\right)^{\frac{1}{2}}\right)^{-1}\left(\int_{|x|=1}|F_{\psi\psi}|^2\right)^{\frac{1}{2}}\left(\int_{|x|=1}|F_{r\psi}|^2\right)^{\frac{1}{2}}.$$

Rearranging this inequality, we have

$$
\begin{aligned}
&\left(2-\sqrt{k}K\Big(\int_{|x|\le 2}|F_A|^2\Big)^{\frac{1}{2}}\right)\int_{|x|\le 1}|F_A|^2\\
&\qquad\le\left(2-\sqrt{k}K\Big(\int_{|x|\le 2}|F_A|^2\Big)^{\frac{1}{2}}\right)\frac{2\sqrt{k}}{\lambda_4}\left(\int_{|x|\le 2}|F_A|^2\right)^{\frac{1}{2}}\int_{|x|\le 1}|F_A|^2\\
&\qquad\qquad+\left(\int_{|x|=1}|F_{\psi\psi}|^2\right)^{\frac{1}{2}}\left(\int_{|x|=1}|F_{r\psi}|^2\right)^{\frac{1}{2}}\\
&\left(2-\sqrt{k}K\Big(\int_{|x|\le 2}|F_A|^2\Big)^{\frac{1}{2}}\right)\left(1-\frac{2\sqrt{k}}{\lambda_4}\Big(\int_{|x|\le 2}|F_A|^2\Big)^{\frac{1}{2}}\right)\int_{|x|\le 1}|F_A|^2\\
&\qquad\le\left(\int_{|x|=1}|F_{\psi\psi}|^2\right)^{\frac{1}{2}}\left(\int_{|x|=1}|F_{r\psi}|^2\right)^{\frac{1}{2}}\\
&\qquad\le\frac{1}{2}\int_{|x|=1}|F_A|^2,
\end{aligned}
$$

where the last inequality comes from the inequality of arithmetic and geometric means, $\sqrt{a}\sqrt{b}\le\frac{1}{2}(a+b)$. By expanding the term on the left, we see that

$$
\left(1-\omega\Big(\int_{|x|\le 2}|F_A|^2\Big)^2\right)\int_{|x|\le 1}|F_A|^2\le\frac{1}{4}\int_{|x|=1}|F_A|^2,
$$

where $\omega=\sqrt{k}(\frac{2}{\lambda_4}+\frac{K\sqrt{k}}{2})$, and this proves the result for $r=1$, which gives the result for any r by dilation. □

Claim 4.4.7. *Let $n=4$. Then, if d_A is a Yang–Mills connection in $B(2,0)-\{0\}$ satisfying $\int_{|x|\le 2}|F_A|^2\le\varepsilon^2$, then $||F_A||_{L^\infty}$ is bounded in $|x|\le 2$.*

Proof. Let ε be the same as in Claim 4.4.6, and assume in addition that $1-\omega\varepsilon=\gamma>0$. Then we have

$$
(1-\omega\varepsilon)\int_{|x|\le r}|F_A|^2\le\frac{r}{4}\int_{|x|=r}|F_A|^2.
$$

Let $f(r)=\int_{|x|\le r}|F_A|^2$, then $f'(r)=\int_{|x|=r}|F_A|^2$, and

$$
\frac{f'(r)}{f(r)}\ge\frac{4(1-\omega\varepsilon)}{r}.
$$

If we then integrate between r and 1, we get

$$\begin{aligned}\int_{|x|=r}^{|x|=1} \frac{4(1-\omega\varepsilon)}{r}dr &\leq \int_{|x|=r}^{|x|=1} \frac{f'(r)}{f(r)}dr \\ -4(1-\omega\varepsilon)\ln r &\leq \ln\frac{f(1)}{f(r)} \\ \ln r^{-4(1-\omega\varepsilon)} &\leq \ln\frac{f(1)}{f(r)} \\ f(r) \leq r^{4(1-\omega\varepsilon)} f(1).\end{aligned}$$

Since $f(1) \leq f(2) \leq \varepsilon^2$, we have

$$\int_{|x|\leq r} |F_A|^2 \leq r^{4\gamma}\varepsilon^2.$$

With this new bound on $f(r)$, we may then apply Claim 4.4.6 again to get

$$4(1-\omega(rd)^{2\gamma}\varepsilon)f(r) \leq rf'(r),$$

which then integrate again to find

$$f(r) \leq r^4 \exp\left(\frac{4\omega\varepsilon}{\gamma}\right) f(1)$$

for $\gamma \leq \frac{1}{2}$. To complete the proof, observe that by Claim 4.4.4, we have

$$\begin{aligned}|F(x)|^2 \leq r^{-4}k\int_{B(0,2r)} |F_A|^2 &= r^{-4}kf(2r) \\ &\leq k2^4 \exp\left(\frac{4\omega\varepsilon}{\gamma}\right) f(1),\end{aligned}$$

which gives a bound of $||F_A||_{L^\infty}$, as claimed. □

Claim 4.4.8. *Let d_A be a Yang–Mills connection on $B^4 - \{0\} = \{x \in \mathbb{R}^4 : 0 < |x| \leq 1\}$ and assume that $||F_A||^q_{L^q(B^n)} < \infty$ for $q \geq 2$. Then there exists a gauge in which the bundle E extends smoothly to $\bar{E}$ over $x = 0$ and d_A extends to a smooth $d_{\bar{A}}$ in $\bar{E}$ which is Yang–Mills.*

Proof. In the case that $q > 2$ we may apply Theorem 4.3.9, and in the case that $q = 2$ we apply Claim 4.4.3 and then Claim 4.4.7 to show uniform boundedness. By Claim 4.4.3 we may assume that $||F_A||_{L^\infty(B^4)} < \gamma_1$, where γ_1 is the constant of Theorem 4.2.8. We may then apply Theorem 4.2.8 which yields a new gauge, $d_{\bar{A}}$ in which we have $d^*\bar{A} = 0$ in B^n. The smoothness is given by Theorem 5.3 of [38]. Since d_A is Yang–Mills over $B^n - \{0\}$, $d_{\bar{A}}$ is Yang–Mills over B^n. □

This concludes the proof of the removable singularities theorem in dimension 4. □

Remark 4.4.9. Note that although we have explicitly constructed a bound for $||F_A||_{L^\infty}$, by Theorem 4.3.9, all we needed to show was that the curvature had growth $|F_A| \sim \dfrac{1}{|x|^{2-\varepsilon}}$ for any $\varepsilon > 0$ to show that the curvature is actually bounded.

Since Uhlenbeck published her removable singularities result there has been a significant improvement, in that the hypothesis of the theorem only requires finite energy of the the connection and does not require that the connection be Yang–Mills. This was published by Uhlenbeck as Theorem 2.1 in [55], although a more modern proof can be found as Theorem 6.2 in [40]. The removability of singularities is important in the construction of the long time solution to the Yang–Mills heat flow problem, and we give a brief discussion on this at the end of the next chapter.

Chapter 5

Yang–Mills Heat Flow Over Real Four-Manifolds

5.1 Introduction

In this chapter we explore the heat flow method used to construct weak solutions of the Yang–Mills equations. Heat flow methods are somewhat classical in differential geometry, and applying this method to the Yang–Mills equations was first suggested by Atiyah and Bott in [1]. Recall that

$$\mathcal{YM}(D) = \frac{1}{2}\int_M |F_D|^2 * (1)$$

is the Yang–Mills functional, where M is again a smooth, finite dimensional, compact, boundary-free, orientable Riemannian manifold. The Euler-Lagrange equation of this functional is given by

$$D^* F_D = 0,$$

which is a non-elliptic system of equations. The heat flow approach aims to deform a connection along the path of steepest descent of the functional in the hope that it will converge to a limiting Yang–Mills connection. Namely, we have the following initial value problem:

$$\begin{cases} \dfrac{d}{dt}D = -D^* F_D \\ D(0) = D_0. \end{cases} \tag{YMHF}$$

Analogously to how $\mathcal{YM}_p(\cdot)$ is not an elliptic operator as discussed in Chapter 3, this problem is non-parabolic because of the gauge invariance. As such, the equation doesn't exhibit the smoothing properties of homogeneous parabolic equations which we've come to know and love. This makes the analysis of the equations that much more complicated in that it raises significant questions around existence and regularity of solutions. Moreover, the solutions could *concentrate*, which prohibits the convergence of the flow to a limiting Yang–Mills connection.

Another major challenge for the analysis of these equations is that the system must be treated globally - it is not possible to analyse the system on co-ordinate patches of M and then patch it together, as we have done in previous chapters. For the reasons discussed in Section 2.9 we must always treat the system globally. Note the difference between this case and the previous chapters - in Chapter 1 and 2 we denoted $D = d_A$ globally by an abuse of notation, because we were not trying to minimise D, but rather trying to find more convenient representatives of a choice of connection. This is different to our goal in this section where we try to find a global minimiser of $\mathcal{YM}(D)$ by deforming D. For this reason we refrain from our previous abuse of notation to emphasise that a connection is a necessarily global object.

The Yang–Mills problem has been treated extensively under various different hypotheses, most notably by Donaldson in [14], where he solves the problem in the smooth case in the critical dimension, and by Råde, who considered the problem under weaker hypotheses in dimensions ≤ 3 in [43]. It is Struwe's 1994 paper, [48], which first addresses the problem over real four manifolds under the weakest possible regularity assumptions. Struwe's paper establishes short-time existence and , but does not address the problem of long-term behaviour, and it is his student, Schlatter, in his 1995 PhD dissertation, [45], which establishes the long term behaviour of the flow. In this chapter we will follow Struwe's original proof of the short term existence and uniqueness of the Yang–Mills heat flow, although we provide commentry on alternative approaches to various different steps in the proof.

5.2 Statement of the Theorem

Before we state the theorem, however, we must introduce the notion of a weak solution of the YMHF. Let (E, π, M) be a vector bundle over M, a smooth Riemannian manifold of dimension four.

Definition 5.2.1. A family $D = D(t)$ of connections on E is a weak solution of the YMHF if $D = D_{\text{ref}} + A(t)$ with

$$A \in L^1\bigg([0,T) : L^2\big(\Omega^1(\text{ad } E)\big)\bigg),$$
$$F_D \in L^\infty\bigg([0,T) : L^2\big(\Omega^2(\text{ad } E)\big)\bigg),$$

and if for any $\phi \in C^\infty\big([0,T] : \Omega^1(\text{ad } E)\big)$ vanishing near $t = 0$ and $t = T$ there holds

$$\int_0^T \left\{\left(A, \frac{d}{dt}\phi\right) - (F_D, D\phi)\right\} dt = 0.$$

Remark 5.2.2. As Struwe notes, this is the weakest possible notion of a solution. This is because in order to interpret $F_D = D \circ D$ in the distributional sense, we require that $A \in L^2$ in space almost everywhere. Furthermore, since the term $(F_D, D\phi)$ involves the product of F and A, we require that $F_D \in L^q(L^2)$ and $A \in L^p(L^2)$, where $p, q \in \mathbb{R}$ are conjugate exponents. Note that have used the notation which we introduce in equation (5.4).

Although this is the weakest possible notion of a solution of the YMHF, Struwe is able to attain a stronger solution in the following:

Theorem 5.2.3. *For any connection D_0 of class H^1 on E such that $\mathcal{YM}(D_0) < +\infty$, there is a $T > 0$ and a weak solution $D = D_{ref} + A$ to the YMHF for*

$0 \leq t < T$ such that

$$A \in C^0\Big([0,T) : L^2\big(\Omega^1(ad\ E)\big)\Big) \cap H^1\Big([0,T) : L^2\big(\Omega^1(ad\ E)\big)\Big),$$
$$F_D \in C^0\Big([0,T) : L^2\big(\Omega^2(ad\ E)\big)\Big).$$

Moreover, D is gauge equivalent to a smooth solution of the YMHF in the following sense: There is a solution $\hat{D} = D_{ref} + \hat{A}$ of the YMHF with

$$\hat{A} \in H^1\Big([0,T) : L^2\big(\Omega^1(ad\ E)\big)\Big) \cap C^0\Big([0,T) : L^2\big(\Omega^1(ad\ E)\big)\Big)$$

and smooth for $0 < t < T$, and a sequence of smooth gauge transformations $\hat{S}_k \in \mathcal{G}$ and a sequence $t_k \searrow 0$ such that $\hat{S}_k \to \hat{S}_0$ in H^1, $\hat{S}_k^*\big(\hat{D}(t_k)\big) \to D_0$ in H^1, and $D = \hat{S}_0^*(\hat{D})$. D is smooth if D_0 is smooth. Furthermore:

(i) If D is irreducible in the sense of (5.25) for all t, then D is unique.

(ii) The maximal existence time T is characterised by

$$T = \left\{ \bar{t} > 0 : \exists\, R > 0 : \sup_{\substack{x_0 \in M \\ 0 \leq t \leq \bar{t}}} \left(\int_{B_R(x_0)} |F_{D(t)}|^2 * (1) \right) < \varepsilon_0 \right\} \tag{5.1}$$

where $\varepsilon_0 = \varepsilon_0(E) > 0$, At $\bar{t}_1 = T$, the curvature concentrates in at most finitely many points $\bar{x}_1^j$, where $j = 1, ..., J_1$ in the sense that for all $R > 0$

$$\limsup_{t \nearrow \bar{t}_1} \int_{B_R(x_1^j)} |F_{D(t)}|^2 * (1) \geq \varepsilon_0.$$

There is clearly a lot to unpack in this theorem, and this whole chapter is devoted to its proof. We first introduce some preliminary estimates in section 5.3 before proving the local existence to the flow in section 5.4. We then continue to consider gauge equivalent solutions of the flow in section 5.5. In section 5.6 we consider the problem of uniqueness before characterising the maximal existence time in section 5.7.

5.3 Some Preliminary Estimates

In this section we give several definitions and results which we will constantly refer back to during the rest of the paper. Our first estimate is a Calderon-Zygmund type inequality which comes as a consequence of the Weizenböck formula, equation (2.5).

Lemma 5.3.1. *Let $D = D_{ref} + A$, $A \in C^1(\Omega^1(ad\ E))$. There exist constants $C_1 = C_1(E)$, $C_2 = C_2(E, ||A||_{C^1})$ such that for any $\phi \in H^2(\Omega^i(ad\ E))$ there holds*

$$||\phi||_{H^2}^2 \leq C_1 ||\Delta_D \phi||_{L^2}^2 + C_2 ||\phi||_{L^2}^2 \tag{5.2}$$

Proof. Firstly, by the Weitzenböck identity, equation (2.5), we have

$$||\Delta_D \phi||_{L^2}^2 = ||\nabla^* \nabla \phi + F \# \phi + \mathrm{Rm} \# \phi||_{L^2}^2.$$

Suppose first that $A = 0$, $D = D_{\text{ref}}$, $\Delta = \Delta_{\text{ref}}$. By Minkowski's inequality, we then have

$$||\Delta_D \phi||_{L^2}^2 \geq ||\nabla^* \nabla \phi||_{L^2}^2 - C||\phi||_{L^2}^2,$$

where $C = ||F||_{L^\infty} + ||\text{Rm}||_{L^\infty} = C(E)$. Note that by the definition of the covariant derivative and its L^2 dual (see, for instance Appendix II of [32]), we have

$$\nabla^* \nabla \phi = \nabla \nabla^* \phi + F_D \# \phi + \text{Rm} \# \phi,$$

and so interchanging the order of derivatives introduces further curvature (both of the connection and of the underlying manifold) terms. With this in mind, note that

$$\begin{aligned}(\nabla^* \nabla \phi, \nabla^* \nabla \phi) &= (\nabla \phi, \nabla \nabla^* \nabla \phi) \\ &= (\nabla \phi, \nabla^* \nabla \nabla \phi + F_D \# \nabla \phi + \text{Rm} \# \nabla \phi) \\ &= (\nabla \nabla \phi, \nabla \nabla \phi) + (\nabla \phi, F_D \# \nabla \phi + \text{Rm} \# \nabla \phi),\end{aligned}$$

and so

$$\begin{aligned}||\nabla^* \nabla \phi||_{L^2}^2 + (\nabla \phi, F_D \# \nabla \phi + \text{Rm} \# \nabla \phi) &= ||\nabla^2 \phi||_{L^2}^2 \\ ||\nabla^* \nabla \phi||_{L^2}^2 + C(E)||\nabla \phi||_{L^2}^2 &\geq ||\nabla^2 \phi||_{L^2}^2 \\ ||\nabla^* \nabla \phi||_{L^2}^2 &\geq ||\nabla^2 \phi||_{L^2}^2 - C(E)||\nabla \phi||_{L^2}^2\end{aligned}$$

By the Gagliardo-Nirenberg interpolation inequality (see, for example, Chapter 9 of [10]) and Young's inequality with $\varepsilon = \frac{1}{2C}$, we have for $n = 4$ that

$$||\nabla \phi||_{L^2} \leq C||\nabla^2 \phi||_{L^2}^{\frac{1}{2}} ||\phi||_{L^2}^{\frac{1}{2}} \leq \frac{1}{4}||\nabla^2 \phi||_{L^2} + C||\phi||_{L^2}.$$

Therefore, we have

$$||\nabla^* \nabla \phi||_{L^2}^2 \geq \frac{1}{2}||\nabla^2 \phi||_{L^2}^2 - C(E)||\phi||_{L^2}^2,$$

and the claimed inequality follows, since we then have

$$||\Delta_D \phi||_{L^2}^2 \geq \frac{1}{2}||\nabla^2 \phi||_{L^2}^2 - C(E)||\phi||_{L^2}^2.$$

In the general case, we have

$$\begin{aligned}\Delta_D \phi &= [(D_{\text{ref}} + A)^*(D_{\text{ref}} + A) + (D_{\text{ref}} + A)(D_{\text{ref}} + A)^*]\phi \\ &= (D_{\text{ref}} + A)^*(D_{\text{ref}} \phi + A \# \phi) + (D_{\text{ref}} + A)(D_{\text{ref}}^* \phi + A \# \phi) \\ &= \Delta_{\text{ref}} \phi + \nabla_{\text{ref}} A \# \phi + A \# \nabla_{\text{ref}} \phi + A \# A \# \phi.\end{aligned} \tag{5.3}$$

Since by the previous calculation we have

$$||\phi||_{H^2}^2 \leq C_1||\Delta_{\text{ref}} \phi||_{L^2}^2 + C_2||\phi||_{L^2}^2.$$

With this in mind, we see that

$$\begin{aligned}||\Delta_D\phi||^2_{L^2} &\geq ||\Delta_{\text{ref}}\phi||^2_{L^2} - ||A\#\nabla_{\text{ref}}\phi||^2_{L^2} - ||\nabla_{\text{ref}}A\#\phi||^2_{L^2} - ||A\#A\#\phi||^2_{L^2} \\ &\geq C(E)^{-1}||\phi||^2_{H^2} - C(E, ||A||_{L^\infty}, ||\nabla A||_{L^\infty})||\phi||^2_{L^2}\end{aligned}$$

and the claimed inequality holds in general. □

We now introduce some notation which will be used throughout the chapter. We denote

$$L^\ell(H^m) := L^\ell\Big([0,T]; H^m\big(\Omega^i(\text{ad }E)\big)\Big). \tag{5.4}$$

Moreover, we use double indices to denote the space-time $L^p - L^q$ norms.

$$||\phi||_{L^{q,p}} := \left(\int_0^T ||\phi(t)||^q_{L^p}\, dt\right)^{1/q}, \quad 1 \leq p, q < \infty.$$

In particular, $||\cdot||_{L^{2,2}}$ denotes the L^2-norm over space-time. For ease of notation, for any $T > 0$ we introduce the space

$$V = V_T(\Omega^i(\text{ad }E)) = L^2\Big([0,T]; H^2\big(\Omega^i(\text{ad }E)\big)\Big) \cap H^1\Big([0,T]; L^2\big(\Omega^i(\text{ad }E)\big)\Big).$$

Note that since

$$\frac{1}{2}\frac{d}{dt}(\phi,\phi) = \left(\frac{d}{dt}\phi, \phi\right) \leq \left|\left|\frac{d}{dt}\phi\right|\right|_{L^2}||\phi||_{L^2} \leq \frac{1}{2}\left(\left|\left|\frac{d}{dt}\phi\right|\right|^2_{L^2} + ||\phi||^2_{L^2}\right)$$

and for $\nabla = \nabla_{\text{ref}}$, since the covariant and time derivatives commute, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\nabla\phi, \nabla\phi) &= \Big(\nabla\frac{d}{dt}\phi, \nabla\phi\Big) = \Big(\frac{d}{dt}\phi, \nabla^*\nabla\phi\Big) \\ &\leq \left|\left|\frac{d}{dt}\phi\right|\right|_{L^2}||\nabla^2\phi||_{L^2} \leq \frac{1}{2}\left(\left|\left|\frac{d}{dt}\phi\right|\right|^2_{L^2} + ||\nabla^2\phi||^2_{L^2}\right).\end{aligned}$$

We have the continuous embedding $V \hookrightarrow L^\infty\Big([0,T]; H^1\big(\Omega^i(\text{ad }E)\big)\Big)$ with

$$\sup_{0\leq t\leq T} ||\phi(t)||^2_{H^1} \leq ||\phi(0)||^2_{H^1} + 2||\phi||^2_V, \tag{5.5}$$

where we have used

$$||\phi||^2_V := \left|\left|\frac{d}{dt}\phi\right|\right|^2_{L^{2,2}} + ||\phi||^2_{L^2(H^2)}.$$

We have in fact that $V \hookrightarrow C^0\Big([0,T]; H^1\big(\Omega^i(\text{ad }E)\big)\Big)$, although we refer to Theorem 3.1 of [33] for a proof.

Lemma 5.3.2. *Let $D = D_{\mathit{ref}} + A$, $A \in C^1(\Omega^1(\mathit{ad}\ E))$. Then there exists a constant $C_2 = C_2(E)$ and a number $T = T(E, A) > 0$ such that for any $\phi \in V_T$ there* *holds*

$$||\phi||_V^2 \leq C_2 \Big|\Big| \Big(\frac{d}{dt} + \Delta_D\Big)\phi \Big|\Big|^2_{L^{2,2}} + C_2 ||\phi(0)||_{H^1};$$

Proof. By expanding $\Big(\Big(\frac{d}{dt} + \Delta_D\Big)\phi, \Big(\frac{d}{dt} + \Delta_D\Big)\phi\Big)$, we get

$$\Big|\Big| \Big(\frac{d}{dt} + \Delta_D\Big)\phi \Big|\Big|^2_{L^{2,2}} = \Big|\Big| \frac{d}{dt}\phi \Big|\Big|^2_{L^{2,2}} + ||\Delta_D \phi||^2_{L^2} + 2\Big(\frac{d}{dt}\phi, \Delta_D \phi\Big)$$

for almost every t. By Lemma 5.3.1 we have

$$||\Delta_D \phi||^2_{L^2} \geq C(E)^{-1} ||\phi||^2_{H^2} - C(E, ||A||_{C^1}) ||\phi||^2_{L^2}.$$

Moreover, by the Weizenböck identity, equation 2.5, and (5.3) we have

$$\begin{aligned} 2\Big(\frac{d}{dt}\phi, \Delta_D \phi\Big) &\geq 2\Big(\frac{d}{dt}\phi, \Delta_{\text{ref}} \phi\Big) - C(E, ||A||_{C^1}) \Big|\Big| \frac{d}{dt}\phi \Big|\Big|_{L^2} ||\phi||_{H^1} \\ &\geq 2\Big(\frac{d}{dt}\phi, \nabla^*_{\text{ref}} \nabla_{\text{ref}} \phi\Big) - C(E, ||A||_{C^1}) \Big|\Big| \frac{d}{dt}\phi \Big|\Big|_{L^2} ||\phi||_{H^1} \\ &= \frac{d}{dt} ||\nabla_{\text{ref}} \phi||^2_{L^2} - C(E, ||A||_{C^1}) \Big|\Big| \frac{d}{dt}\phi \Big|\Big|_{L^2} ||\phi||_{H^1} \\ &\geq \frac{d}{dt} ||\nabla_{\text{ref}} \phi||^2_{L^2} - \frac{1}{2} \Big|\Big| \frac{d}{dt}\phi \Big|\Big|^2_{L^2} - C(E, ||A||_{C^1}) ||\phi||^2_{H^1}, \end{aligned}$$

Where the last inequality was obtained by Young's inequality with $\varepsilon = \frac{1}{C(E, ||A||_{C^1})}$. Putting this all together, we find

$$\begin{aligned} \Big|\Big| \Big(\frac{d}{dt} + \Delta_D\Big)\phi \Big|\Big|^2_{L^{2,2}} \geq C(E)^{-1} ||\phi||^2_{H^2} + \frac{1}{2} \Big|\Big| \frac{d}{dt}\phi \Big|\Big|^2_{L^2} \\ + \frac{d}{dt} ||\nabla_{\text{ref}} \phi||^2_{L^2} - C(E, ||A||_{C^1}) ||\phi||^2_{H^1} \end{aligned}$$

Then by integrating in time, we obtain

$$||\phi||_V^2 \leq C(E) \left(\Big|\Big| \Big(\frac{d}{dt} + \Delta_D\Big)\phi \Big|\Big|^2_{L^{2,2}} + ||\phi(0)||^2_{H^1} \right) + C(E, ||A||_{C^1}) ||\phi||^2_{L^2(H^1)}.$$

Finally, we use (5.5) to estimate

$$||\phi(t)||^2_{L^2(H^1)} \leq T ||\phi||^2_{L^\infty(H^1)} \leq T ||\phi(0)||^2_{H^1} + 2T ||\phi||^2_V.$$

If we choose $T = \frac{1}{4(1 + C(E, ||A||^1_C))}$, then the lemma follows. □

The linear estimates above don't necessarily hold for borderline case $A \in H^1$; however, under certain circumstances L^p-estimates are still available. To obtain these estimates we first observe that for any $\phi, \psi \in \Omega^i(\text{ad } E)$, and since we are assuming D is a metric connection, we have

$$d\langle \phi, \psi \rangle = \langle D\phi, \psi \rangle + \langle \phi, D\psi \rangle$$

In particular, for sections $\phi \in \Omega^0(\text{ad } E)$, where $\nabla\phi = D\phi$, we obtain Kato's inequality

$$|d|\phi|| \leq |D\phi|.$$

We can combine this with the Sobolev embedding $H^1 \hookrightarrow L^4$ valid in 4 dimensions to obtain

$$||\phi||_{L^4} \leq C(||D\phi||_{L^2} + ||\phi||_{L^2})$$

for any $\phi \in \Omega^0(\text{ad } E)$ with a uniform constant $C = C(E)$ independent of D.

To obtain similar estimates for forms of degree $i \geq 1$ we need to consider the exterior covariant derivatives and the covariant derivatives. Note that on account of the Weitzenböck formula this introduces an extra curvature term, but these can be accounted for by constants so long as the curvature doesn't concentrate. Towards these ends we have the following:

Lemma 5.3.3. *Let $D = D_{ref} + A$, $A \in H^1$, with curvature $F = F_D \in L^2$. There exist constants $C_3 = C_3(E)$, $\delta = \delta(E) > 0$ such that for any $\phi \in \Omega^i(\text{ad } E)$, any $0 < R < 1$, there holds*

$$||\phi||^2_{L^4} + ||\nabla\phi||^2_{L^2} \leq C_3\Big(||D\phi||^2_{L^2} + ||D^*\phi||^2_{L^2}\Big) + C_3 R^{-2}||\phi||^2_{L^2},$$

provided

$$\sup_{x_0} \int_{B_R(x_0)} |F_D|^2 * (1) \leq \delta.$$

Proof. By the Weitzenböck identity, equation (2.5), and Sobolev's embedding theorem, we find

$$\begin{aligned} C(E)^{-1}||\phi||^2_{L^4} - ||\phi||^2_{L^2} \leq ||\nabla\phi||^2_{L^2} = (\nabla^*\nabla\phi, \phi) &= (\Delta_D\phi, \phi) \\ &\quad + (F_D\#\phi, \phi) + (\text{Rm}\#\phi, \phi) \\ &= ||D\phi||^2_{L^2} + ||D^*\phi||^2_{L^2} + (F_D\#\phi, \phi) + (\text{Rm}\#\phi, \phi). \end{aligned}$$

Now, by Hölder's inequality, we have

$$(\text{Rm}\#\phi, \phi) \leq C||\phi|||^2_{L^2}$$

To estimate the term $(F\#\phi, \phi)$ we use the Gagliardo-Nirenberg-Sobolev inequality on a suitable cover of M by balls $B_R(x_i)$. To do this we adopt a cutoff function argument. Let $\varphi_i \in C_0^\infty(B_R(x_i))$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{\frac{R}{2}}(x_i)$ and $|\nabla\varphi_i| \leq CR^{-1}$, where $x_i \in M$. We also utilise the Sobolev embedding $H^1 \hookrightarrow L^4$ valid in dimension 4.

$$\begin{aligned} (F_D\#\phi, \phi) &\leq C\sum_i ||F_D||_{L^2(B_R(x_i))}||\varphi_i\phi||^2_{L^4(B_R(x_i))} \\ &\leq C\delta\sum_i \big(||\nabla\phi||^2_{L^2(B_R(x_i))} + R^{-2}||\phi||^2_{L^2(B_R(x_i))}\big), \end{aligned}$$

where $C = C(E)$. Since M is compact, there exists a constant $R_0 = R_0(M) > 0$ and a number L (independent of M) such that for $0 < R \leq R_0 < 1$ there is a cover $(B_R(x_i))$ such that at most L distinct balls of this cover overlap at any point of M.

Thus, for $R \leq R_0$ and with this choice of $(B_R(x_i))$ the above estimate yields

$$(F_D\#\phi, \phi) \leq CL\delta\big(||\nabla\phi||^2_{L^2} + R^{-2}||\phi||^2_{L^2}\big).$$

If we choose $\delta \leq \frac{1}{2CL}$, then we see that

$$||\nabla\phi||^2_{L^2} \leq \frac{1}{2}\left(C_3\Big(||D\phi||^2_{L^2} + ||D^*\phi||^2_{L^2}\Big) + C_3R^{-2}||\phi||^2_{L^2}\right),$$

which implies

$$||\phi||^2_{L^4} \leq \frac{1}{2}\left(C_3\Big(||D\phi||^2_{L^2} + ||D^*\phi||^2_{L^2}\Big) + C_3R^{-2}||\phi||^2_{L^2}\right),$$

which yields the result. □

5.3.1 Evolution of Curvature and Energy Inequality

If we consider the evolution of curvature as $D = D_{\text{ref}} + A(t)$, then for $\sigma \in \Omega^0(\text{ad } E)$ we find

$$\begin{aligned}
\frac{d}{dt}F_D(\sigma) &= \lim_{\varepsilon\to 0}\frac{F_{D+\varepsilon\frac{d}{dt}D}(\sigma) - F_D(\sigma)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{(D+\varepsilon\frac{d}{dt}D)(D\sigma + \varepsilon\frac{d}{dt}D\sigma) - D^2(\sigma)}{\varepsilon}\\
&= \Big(\frac{d}{dt}A(t)\Big)(D\sigma) + D\Big(\frac{d}{dt}(A(t)\Big)\sigma - \frac{d}{dt}A(t)(D\sigma)\\
&= D\Big(\frac{d}{dt}D\Big)(\sigma).
\end{aligned}$$

Therefore, since we are evolving the curvature along the lines of steepest descent for the functional, we find that

$$\frac{d}{dt}F_{D(t)} = -DD^*F_{D(t)}. \tag{5.6}$$

This equation is non-parabolic, and the culprit is again the infinite dimensional gauge group. In view of the first Bianchi identity though, we find that F satisfies the evolution equation

$$\left(\frac{d}{dt} + \Delta_D\right)F_D = 0. \tag{5.7}$$

By taking the inner product of (5.7) with F_D and recalling the second Bianchi identity, we obtain the identity

$$0 = \frac{1}{2}\frac{d}{dt}||F_D||_{L^2}^2 + (\Delta_D F_D, F_D) = \frac{d}{dt}\mathcal{YM}(D) + ||D^* F_D||_{L^2}^2.$$

In particular, by integrating in time we find that any classical solution of the YMHF problem must satisfy for any $T > 0$

$$\mathcal{YM}(D(T)) + \int_0^T ||D^* F_D||_{L^2}^2 dt \leq \mathcal{YM}(D_0). \tag{5.8}$$

Moreover, if we multiply F_D by $F_D\varphi^2$, where φ is a cutoff function with support in $B_{2R}(x_0)$ such that $\varphi \equiv 1$ on $B_R(x_0)$ and $|\nabla\varphi| \leq CR^{-1}$, as before, we find

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\int_{B_{2R}(x_0)} |F_D|^2\varphi^2 * (1) + (\Delta F_D, F_D\varphi^2) \\
&= \frac{1}{2}\frac{d}{dt}\int_{B_{2R}(x_0)} |F_D|^2\varphi^2 * (1) + (DD^* F_D, F_D\varphi^2) \\
&= \frac{1}{2}\frac{d}{dt}\int_{B_{2R}(x_0)} |F_D|^2\varphi^2 * (1) + (D^* F_D, D^* F_D\varphi^2) + 2(D^* F_D, F_D\varphi D^*\varphi).
\end{aligned}$$

Then since $(\cdot,\cdot)$ is the L^2 inner product introduced in Section 2.7, we have by the Cauchy-Schwarz and Young's inequalities that

$$2(D^* F_D, F_D\varphi D^*\varphi) \leq \int_{B_{2R}(x_0)} |D^* F_D|^2\varphi^2 * (1) + \int_{B_{2R}(x_0)} |F_D|^2|\nabla\varphi|^2 * (1).$$

If we re-arrange what we currently have, we write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{B_{2R}(x_0)} |F_D|^2\varphi^2 * (1) + \int_{B_{2R}(x_0)} |D^* F_D|^2\varphi^2 * (1) &= -2(D^* F_D, F_D\varphi D^*\varphi) \\
&\leq \int_{B_{2R}(x_0)} |D^* F_D|^2\varphi^2 * (1) + \int_{B_{2R}(x_0)} |F_D|^2|\nabla\varphi|^2 * (1),
\end{aligned}$$

which, if we then integrate in time we find that

$$\int_{B_R(x_0)} |F_{D(t)}|^2 * (1) \leq \int_{B_{2R}(x_0)} |F_{D_0}|^2 * (1) + CtR^{-2}\mathcal{YM}(D_0),$$

since $\mathcal{YM}(D(t)) \leq \mathcal{YM}(D_0)$ for all $t \geq 0$. Note that $\mathcal{YM}(D_0)$ has support in the ball $B_{2R(x_0)}$. Therefore, for any $0 \leq t \leq T$, where T is the maximum existence time, we have

$$\sup_{0\leq t\leq T}\int_{B_R(x)} |F_{D(t)}|^2 \ * (1) \leq \int_{B_{2R}(x_0)} |F_{D_0}|^2 * (1) + CTR^{-2}\mathcal{YM}(D_0). \tag{5.9}$$

Next, observe that by Hölder's and Young's inequalities $L^{\infty,2} \cap L^{2,4} \hookrightarrow L^{3,3}$ with

$$||\phi||_{L^{3,3}}^2 \leq ||\phi||_{L^{\infty,2}}^{\frac{2}{3}}||\phi||_{L^{2,4}}^{\frac{4}{3}} \leq \frac{1}{3}||\phi||_{L^{\infty,2}}^2 + \frac{2}{3}||\phi||_{L^{2,4}}^2 \tag{5.10}$$

for any ϕ. Combining this with Lemma 5.3.3, equation (5.8) and the first Bianchi identity, we obtain :

Lemma 5.3.4. *Let $\delta = \delta(E) > 0$ be as in Lemma 5.3.3 and suppose D is a classical solution of YMHF on $(0, T)$ with*

$$\sup_{\substack{x_0 \in M \\ 0<t<\bar{t}}} \int_{B_R(x_0)} |F_{D(t)}|^2 \ * (1) < \delta \tag{5.11}$$

for some $0 < R \leq 1$. Then $F \in L^{3,3}$ with

$$||F_D||^2_{L^{3,3}} \leq C_4(1 + TR^{-2})\mathcal{YM}(D_0),$$

where $C_4 = C(E)$.

Proof. By equation (5.10) we have

$$||F_D||^2_{L^{3,3}} \leq \frac{1}{3}||F_D||_{L^{\infty,2}} + \frac{2}{3}||F_D||^2_{L^{2,4}}.$$

Recall then that by equation (5.9) we have

$$||F_D||_{L^{\infty,2}} \leq 2\mathcal{YM}(D_0) + CTR^{-2}\mathcal{YM}(D_0),$$

and by integrating the result from Lemma 5.3.3 in time, and noting that by equation (5.8)

$$||F_D||^2_{L^{2,4}} \leq \mathcal{YM}(D_0) + CTR^{-2}\mathcal{YM}(D_0),$$

and putting this together yields the result. □

As a consequence of the Weitzenböck identity and equation (5.7) we then have the following lemma:

Lemma 5.3.5. *Under the assumptions of Lemma 5.3.4 there holds*

$$D^*F_D \in L^2_{loc}\Big((0,T]; L^4\big(\Omega^1(ad\ E)\big)\Big), \quad \frac{d}{dt}F_D \in L^2_{loc}\Big((0,T]; L^2\big(\Omega^2(ad\ E)\big)\Big).$$

Proof. By evaluating $\left(\left(\frac{d}{dt} + \Delta_D\right)F_D, \left(\frac{d}{dt} + \Delta_D\right)F_D\right) = 0$ and recalling the first Bianchi identity, we have at any time $t > 0$ that

$$\begin{aligned}
\Big|\Big|\frac{d}{dt}F_D\Big|\Big|^2_{L^2} + ||DD^*F_D||^2_{L^2} &= -2\Big(DD^*F_D, \frac{d}{dt}F_D\Big) \\
&= \Big|\Big|\frac{d}{dt}F_D\Big|\Big|^2_{L^2} + ||DD^*F_D||^2_{L^2} \\
&= -2\Big(DD^*F_D, \frac{d}{dt}F_D\Big) \\
&= -2\Big(D^*F_D, \frac{d}{dt}(D^*F_D)\Big) + 2\Big(D^*F_D, \Big(\frac{d}{dt}D\Big)\#F_D\Big) \\
&\leq -\frac{d}{dt}||D^*F_D||^2_{L^2} + 2||F_D||_{L^3}||D^*F_D||^2_{L^3},
\end{aligned}$$

where the last inequality holds by the fact that $\frac{d}{dt}D = -D^*F_D$ and generalised Hölder's inequality. Since we are under the assumption that (5.11) is true we have that Lemma 5.3.3 applies. From this and the second Bianchi identity there holds

$$||D^*F_D||_{L^4}^2 \le C||DD^*F_D||_{L^2}^2 + CR^{-2}||D^*F_D||_{L^2}^2,$$

for any t. We then have the energy inequality

$$\begin{aligned}\Big|\Big|\frac{d}{dt}F_D\Big|\Big|_{L^2}^2 + ||D^*F_D||_{L^4}^2 + \frac{d}{dt}||D^*F_D||_{L^2}^2 \\ \le C\bigg(2||F_D||_{L^3}||D^*F_D||_{L^3}^2 + R^{-2}|D^*F_D||_{L^2}^2\bigg),\end{aligned}$$

which we may then integrate in time on any interval $[t_0, t_1] \subset [0, T]$ we obtain

$$\begin{aligned}&\Big|\Big|\frac{d}{dt}F_D\Big|\Big|_{L^{2,2}}^2 + ||D^*F_{D(t_1)}||_{L^2}^2 + ||D^*F_D||_{L^{2,4}}^2 \\ &\quad\le C||F_D||_{L^{3,3}}||D^*F_D||_{L^{3,3}}^2 + ||D^*F_{D(t_0)}||_{L^2}^2 + CR^{-2}||D^*F_D||_{L^{2,2}}^2 \\ &\quad\le C_5||F_D||_{L^{3,3}}\Big(||D^*F_D||_{L^{\infty,2}}^2 + ||D^*F_D||_{L^{2,4}}^2\Big) \\ &\qquad\quad + ||D^*F_{D(t_0)}||_{L^2}^2 + C_5(t_1 - t_0)R^{-2}||D^*F_D||_{L^{\infty,2}}^2,\end{aligned}$$

where we also used equation (5.10). To show that the lemma is true on the interval $[t_0, t_1]$ we must show that the right hand side of the above inequality is finite. We are not able to integrate over the entire time domain because we will need to use the fact that we can make t_1 and t_0 close enough together achieve a smallness estimate.

By the mean value theorem, given $\tau > 0$ we can find $t_0 \in [0, \tau]$ such that

$$||D^*F_{D(t_0)}||_{L^2}^2 \le 2\tau^{-1}\int_0^\tau ||D^*F_{D(t)}||_{L^2}^2\, dt \le 2\tau^{-1}\mathcal{YM}(D_0).$$

Moreover, by the absolute continuity of the Lebesgue integral, i.e. that if $t_1 - t_2 = h$ then $\lim_{h\to 0}\int_{t_1}^{t_2} f(x)dx = 0$, and Lemma 5.3.4, we can achieve that

$$||F_D||_{L^{3,3}}^3 = \Big(\int_{t_0}^{t_1} ||F_{D(t)}||_{L^3}^3\, dt\Big) \le \frac{1}{4C_5}$$

uniformly in t_0 and t_1, if the difference $h = t_1 - t_0$ is sufficiently small. We may assume that $h \le \frac{R^2}{4C_5}$.

Finally, for any such pair $t_0 < t_1 < t_0 + h$ we may choose $t_1' \in [t_0, t_1]$ such that

$$||D^*F_{D(t_1')}||_{L^2}^2 \ge \frac{2}{3}||D^*F_D||_{L^{\infty,2}}^2,$$

since we may just choose t_1' such that $\sup_{t_0<t<t_1} ||D^*F_{D(t)}||_{L^2}^2 = ||D^*F_{D(t_1')}||_{L^2}^2$. Hence we obtain the assertion of the lemma on $[t_0, t_1]$. If we cover the interval $[\tau, T]$ with finitely many intervals of length h then the lemma follows. $\square$

Lemma 5.3.6. *Under the assumptions of Lemma 5.3.4 there holds* $D = D_{ref} + A$*, where* A *extends to* $A \in C_{loc}\Big((0,T]; H^1\big(\Omega^1(ad\ E)\big)\Big)$.

Proof. By Lemma 5.3.5 and since $\frac{d}{dt}D = -D^* F_D$, we have that

$$\frac{d}{dt}A \in L^2_{\text{loc}}\big((0,T]; L^4\big),$$

For $n = 1$ we have the Sobolev embedding $H^1 \hookrightarrow C^0$, we see that $A \in C^0\big((0,T]; L^4\big)$.

Thus, since $F_D = F_{D_{\text{ref}}} + D_{\text{ref}}A + A \wedge A$, where $D = D_{\text{ref}} + A$, we see that

$$\frac{d}{dt}(D_{\text{ref}}A) = \frac{d}{dt}F_D + \frac{d}{dt}A\#A \in L^2_{\text{loc}}\big((0,T]; L^2\big),$$

which implies

$$D_{\text{ref}}A \in C^0\big((0,T]; L^2\big).$$

Moreover, by the second Bianchi identity and Lemma 5.3.5, we have

$$\frac{d}{dt}(D^*_{\text{ref}}A) = D^*_{\text{ref}}\Big(\frac{d}{dt}D\Big) = D^*_{\text{ref}}D^*F_D = A\#D^*F_D \in L^2_{\text{loc}}\big((0,T]; L^2\big),$$

which yields that

$$D^*_{\text{ref}}A \in C^0\big((0,T]; L^2\big)$$

and therefore also the claimed result. □

5.4 Local Existence

Although the heat flow method of proof of existence of Yang–Mills connections was suggested by Atiyah and Bott, it was Atiyah's student, Donaldson, in his paper [14] who proved global existence for the Yang–Mills gradient flow for a complex vector bundle. Although such a result was a major breakthrough and has been much celebrated, the hypotheses of this result were very strong, in that all of the gauge transforms and connection forms were assumed to be smooth. This evolution also occurs over a Kähler manifold which introduces extra structure in addition to that of a Riemannian manifold, namely a complex and symplectic structure. Nevertheless, before reviewing Struwe's method for heat flow over a *real* Riemannian manifold of dimension 4 with the weakest possible regularity assumptions on the space of connections and the gauge group, it is instructive to review Donaldson's approach for the smooth case. We will not exactly consider Donaldson's approach, but rather we will consider the same setting as ours, but the case where all maps are smooth. In actual fact, they need not even be smooth for this method to work, but only in the 'safe' Sobolev range where multiplication and inversion of elements are smooth.

First, let $D = D(t)$ be a solution of $\frac{d}{dt}D = -D^*F_D$ and $S = S(t)$ a family of gauge transformations depending smoothly on t. Then $\bar{D} = S^*(D)$.

Then, for $\sigma \in \Omega^0(E)$,

$$\begin{aligned}
\Big(\frac{d}{dt}\bar{D}\Big)\sigma &= \left(\frac{d}{dt}\Big(S^{-1}\circ D\circ S\Big)\right)\sigma \\
&= -\big(S^{-1}\frac{dS}{dt}S^{-1}\circ D\circ S\big)\sigma + \big(S^{-1}\circ\frac{d}{dt}D\circ S\big)\sigma + \left(S^{-1}\circ D\circ\frac{dS}{dt}\right)\sigma \\
&= -s\bar{D}\sigma - \bar{D}^*\bar{F}\sigma + \bar{D}(s\sigma) \\
&= (-\bar{D}^*\bar{F} + \bar{D}s)\sigma,
\end{aligned}$$

where $s = S^{-1}\dfrac{dS}{dt} \in \Omega^0(\text{ad } E)$ and $\bar{F} = F_{\bar{D}}$. Therefore

$$\frac{d}{dt}\bar{D} = -\bar{D}^*\bar{F} + \bar{D}s. \tag{5.12}$$

5.4.1 Donaldson's Ansatz

In the smooth case it is possible to make the ansatz that $s = S^{-1}\dfrac{dS}{dt}$ is of a particular form to produce unique gauge transformations to normalise the gauge equivalent flow. Since we are in the Sobolev range such that the gauge group action is smooth, this will work and we will be able to find a solution to the flow by a gauge transformation. In what is now called the Donaldson–DeTurk trick, Donaldson makes the ansatz that $\bar{D} = D_0 + a$, where he determines a by solving the initial value problem

$$\begin{cases} \dfrac{d}{dt}\bar{D} = \dfrac{d}{dt}a = -\bar{D}^*\bar{F} + \bar{D}(-\bar{D}^*a) \\ a(0) = 0. \end{cases} \tag{5.13}$$

Choosing a such that it is the unique solution of

$$\begin{aligned} S^{-1}\frac{dS}{dt} &= -\bar{D}^*a = s, \\ a(0) &= 0 \end{aligned}$$

actually uniquely determines a gauge transformation such that $\tilde{a} = (S^{-1})^*a$ and $\tilde{D} = (S^{-1})^*\bar{D}$ such that

$$\begin{cases} \dfrac{d}{dt}\tilde{D} + \tilde{D}^*\tilde{F} = 0, \\ \tilde{a}(0) = 0, \end{cases} \tag{5.14}$$

thus solving the YMHF. The proof that this is true is rather long and is not relevant to our argument, so we refer to Lemma 20.3 of [18] or Chapter 4 of [26], among other places, for an exposition of Donaldson's method. The main point to note though, is that such a method fails for the class of connection $A \in H^1$. This is because solving equation 5.13 will only generate a solution $a \in H^1$, which leads to $s \in L^2$ and therefore S defined as the solution of

$$\begin{cases} \dfrac{dS}{dt} = S \circ s \\ S(0) = \mathrm{id}, \end{cases} \tag{5.15}$$

will only be bounded and measurable for $x \in M$, and this is not enough to interpret $\tilde{D} = (S^{-1})^* \bar{D}$ as a weak solution of the YMHF. We may overcome such a difficulty, however, if we do not attempt to fix the background connection, and instead evolve it smoothly. The effect of this will be two-fold: Firstly, evolving the background connection will allow us to obtain one derivative more regularity on s, i.e. $s \in H^1$. Secondly, by evolving the background connection we can view a solution of the YMHF as a small perturbation of D_{bg}. Note that by considering the solution as a small perturbation of D_{bg}, we have that $\bar{D} = D_{\mathrm{bg}} + a$, where $||a||_{L^{\infty,4}} \leq \varepsilon$.

Alternatively, it is also possible to fix a smooth background connection and *impose* the condition $||a(0)||_{L^{\infty,4}} \leq \varepsilon$, which is the method adopted by Feehan in Chapter 19 of his exposition of the Yang–Mills heat flow [18]. Either way, we consider the solution of the YMHF to be a small perturbation of D_{bg}, although one must still evolve the background connection and apply Struwe's argument in Section 5.5. Due to this, we believe it simpler to follow Struwe's original proof and evolve the background connection for regularity reasons as well.

5.4.2 Choice of Background Connection

Let $D_0 \in H^1$ be given and fix a smooth $D_1 \in \Omega^1(\mathrm{ad}\ E)$ and express $D_0 = D_1 + A_0$ in terms of this connection. To determine the evolving background connection, solve the initial value problem

$$\begin{aligned} \frac{d}{dt} A_{\mathrm{bg}} + \Delta_1 A_{\mathrm{bg}} &= 0 \\ A_{\mathrm{bg}}(0) &= A_0, \end{aligned} \tag{5.16}$$

where $\Delta_1 = D_1^* D_1 + D_1 D_1^*$ is the Laplace operator for D_1.

The IVP (5.16) has a unique, global solution A_{bg} since it is the canonical form of the heat equation. Moreover, by the smoothing effect of the homogeneous heat equation, see, for example, Chapter 2.3 of [17], we have that A_{bg} is smooth for $t > 0$. By Lemma 5.3.2 we have $A_{\mathrm{bg}} \in L^2(H^2) \cap C^0(H^1) \cap H^1(L^2)$ with estimates depending on D_1 and A_0 (recall that $V \hookrightarrow C^0\Big([0,T]; H^1(\Omega^i\big(\mathrm{ad}\ E)\big)\Big)$). In particular, by Lemma 5.3.2 since A_{bg} solves the heat equation we have

$$||A_{\mathrm{bg}}||_V \leq C||A_0||_{H^1}.$$

Moreover, by the embedding $V \hookrightarrow C^0\Big([0,T]; H^1(\Omega^i\big(\mathrm{ad}\ E)\big)\Big)$ we have that on the interval $[0, T_1]$ there holds

$$||A_{\mathrm{bg}}||_{L^\infty(H^1)} \leq C||A_0||_{H^1}. \tag{5.17}$$

Moreover, $||A_0||_{H^1}$ may be chosen as small as we like. Finally, we let $D_{\text{bg}} = D_1 + A_{\text{bg}}$ to define our evolving background connection.

5.4.3 Local Existence for the Gauge-Equivalent Flow

We continue to follow Struwe's Ansatz where he sets

$$\begin{aligned} \bar{D} &= D_{\text{bg}} + a, \\ a(0) &= 0, \end{aligned}$$

where $\bar{D}$ solves (5.12) with $s = -\bar{D}^* a$, i.e.

$$\frac{d}{dt}\bar{D} + \bar{D}^*\bar{F} + \bar{D}(\bar{D}^* a) = 0, \tag{5.18}$$

where $\bar{F} = F_{\bar{D}}$. The method, then, is to write $\bar{D} = D_{\text{bg}} + a$, where $||a||_{L^{\infty,4}} \leq \varepsilon$, substitute this into equation (5.18) and then determine if there exists an a which solves this equation. Evolving the background connection is crucial to the argument, since it is this which gives us the fact that $||a||_{L^{\infty,4}} \leq \varepsilon$. Morevoer, the Sobolev embedding for $n = 4$, $H^1 \hookrightarrow L^4$, gives us that $||a||_{L^\infty(H^1)} \leq \varepsilon$, and the following argument would break down if this weren't the case.

By expanding

$$\bar{F} = F_{\text{bg}} + \bar{D}a + a\#a,$$

we get

$$\begin{aligned} \frac{d}{dt}a + \bar{\Delta}a &= -\bar{D}^*(F_{\text{bg}} + a\#a) - \frac{d}{dt}D_{\text{bg}} \\ &= -D^*_{\text{bg}}F_{\text{bg}} - \frac{d}{dt}A_{\text{bg}} + a\#F_{\text{bg}} - \bar{D}^*(a\#a) \end{aligned} \tag{5.19}$$

for a, where $\bar{\Delta}$ is the laplace operator for $\bar{D}$ and $F_{\text{bg}} = F_{D_{\text{bg}}}$. Note that by Lemma 5.3.5 we have

$$\begin{aligned} F_{\text{bg}} &= F_{D_1 + A_{\text{bg}}} \\ &= F_{D_1} + D_1 A_{\text{bg}} + A_{\text{bg}}\#A_{\text{bg}} \in L^2(H^1) \cap C^0(L^2) \\ D^*_{\text{bg}}F_{\text{bg}} &= D_1^* F_{\text{bg}} + A_{\text{bg}}\#F_{\text{bg}} \in L^{2,2} \end{aligned}$$

and are smooth for $t > 0$. Here and in the following we use Sobolev's embedding $H^1 \hookrightarrow L^4$ and Hölder's inequality.
Moreover, $\bar{D}^*(a\#a) = \big(\nabla_{\text{bg}}a\big)\#a + a\#a\#a$. Therefore, we see that a satisfies

$$\frac{d}{dt}a + \bar{\Delta}a = f + a\#F_{\text{bg}} + \nabla_{\text{bg}}a\#a + a\#a\#a,$$

where $f = -D_1^* F_{\text{bg}} + A_{\text{bg}}\#F_{\text{bg}} - \frac{d}{dt}A_{\text{bg}} \in L^{2,2}$ and is smooth for $t > 0$.

Moreover, since

$$\bar{\Delta}\phi = \Delta_1\phi + \nabla_1 A\#\phi + A\#\nabla_1\phi + A\#A\#\phi.$$

where $\bar{A} = A_{\text{bg}} + a$ and $\phi \in H^2(\Omega^i(\text{ad } E))$, we obtain

$$\begin{aligned}\frac{d}{dt}a + \Delta_1 a = f + &F_{\text{bg}}\#a + A_{\text{bg}}\#\nabla_1 a + \nabla_1 A_{\text{bg}}\#a \\ &+ A_{\text{bg}}\#A_{\text{bg}}\#a + a\#\nabla_1 a + A_{\text{bg}}\#a\#a + a\#a\#a.\end{aligned} \tag{5.20}$$

We can now estimate each term in this equation to show that we can use the contraction mapping method to deduce the existence of a solution to this equation, $a \in V_T$. In fact, since $a(0) = 0$, by Lemma 5.3.2 we have

$$||a||_{L^\infty(H^1)} \leq C||a||_V \leq C\Big|\Big|\Big(\frac{d}{dt} + \Delta_1\Big)a\Big|\Big|_{L^{2,2}},$$

on any interval $[0, T]$, where $0 < T \leq T_1$, $C = C(E)$. We may estimate the eight terms on the right of equation (5.20) as follows: By Hölder's inequality and the Sobolev embedding theorem,

$$\begin{aligned}||F_{\text{bg}}\#a||_{L^{2,2}} &\leq ||F_{\text{bg}}||_{L^{2,4}}||a||_{L^{\infty,4}} \leq C||F_{\text{bg}}||_{L^2(H^1)}||a||_{L^\infty(H^1)} \\ &\leq C||A_{\text{bg}}||_{L^2(H^2)}||a||_{L^\infty(H^1)} \leq \varepsilon||a||_{L^\infty(H^1)} \leq \varepsilon||a||_V,\end{aligned}$$

if $0 < T \leq T(\varepsilon, D_1, A_0)$. Since $f \in L^{2,2}$, we may make it arbitrarily small by making the time interval small enough due to the absolute continuity of the Lebesgue integral. We now estimate

$$\begin{aligned}||A_{\text{bg}}\#\nabla_1 a||_{L^{2,2}} &\leq ||A_{\text{bg}}||_{L^{\infty,4}}||\nabla_1 a||_{L^{2,4}} \\ &\leq C||A_0||_{H^1}||a||_{L^2(H^2)} \leq \varepsilon||a||_{L^2(H^2)} \leq \varepsilon||a||_V\end{aligned}$$

If $||A_0||_{H^{1,2}} < \frac{\varepsilon}{C}$, with $C = C(E)$. We now estimate

$$\begin{aligned}||\nabla_1 A_{\text{bg}}\#a||_{L^{2,2}} &\leq ||\nabla_1 A_{\text{bg}}||_{L^{2,4}}||a||_{L^{\infty,4}} \\ &\leq C||A_{\text{bg}}||_{L^2(H^1)}||a||_{L^\infty(H^1)} \leq \varepsilon||a||_V,\end{aligned}$$

and

$$\begin{aligned}||A_{\text{bg}}\#A_{\text{bg}}\#a||_{L^{2,2}} &\leq ||A_{\text{bg}}\#A_{\text{bg}}||_{L^{2,4}}||a||_{L^{\infty,4}} \\ &\leq C||A_{\text{bg}}\#A_{\text{bg}}||_{L^2(H^1)}||a||_{L^\infty(H^1)} \\ &\leq C||a||_{L^\infty(H^1)}\big(||A_{\text{bg}}\#A_{\text{bg}}||_{L^{2,2}} + ||\nabla_1 A_{\text{bg}}\#A_{\text{bg}}||_{L^{2,2}}\big) \\ &\leq C||a||_{L^\infty(H^1)}\big(||A_{\text{bg}}||_{L^{2,4}}||A_{\text{bg}}||_{L^{\infty,4}} + ||\nabla_1 A_{\text{bg}}||_{L^{2,4}}||A_{\text{bg}}||_{L^{\infty,4}}\big) \\ &\leq \varepsilon||a||_V.\end{aligned}$$

We now estimate

$$\begin{aligned}||a\#\nabla_1 a||_{L^{2,2}} &\leq ||a||_{L^{\infty,4}}||\nabla_1 a||_{L^{2,4}} \\ &\leq C||a||_{L^\infty(H^1)}||a||_{L^2(H^2)} \leq \varepsilon||a||_V,\end{aligned}$$

provided that $||a||_{L^\infty(H^1)} \leq \frac{\varepsilon}{C}$. Similarly, we estimate

$$\begin{aligned}||A_{\text{bg}}\#a\#a||_{L^{2,2}} &\leq ||A_{\text{bg}}||_{L^{\infty,4}}||a\#a||_{L^{2,4}} \leq C||A_{\text{bg}}||_{L^\infty(H^1)}||a\#a||_{L^2(H^1)} \\ &\leq C||A_0||_{H^1}\big(||\nabla_{\text{ref}}a\#a||_{L^{2,2}} + ||a||^2_{L^{4,4}}\big) \\ &\leq C||A_0||_{H^1}||a||_{L^{\infty,4}}\big(||\nabla_{\text{ref}}a||_{L^{2,4}} + ||a||_{L^{2,4}}\big) \\ &\leq C||A_0||_{H^1}||a||_{L^\infty(H^1)}||a||_{L^2(H^2)} \leq \varepsilon||a||_V,\end{aligned}$$

provided that $||a||_{L^\infty(H^1)} \leq \frac{\varepsilon}{C}$. Lastly, we estimate

$$||a\#a\#a||_{L^{2,2}} \leq C||a||^2_{L^\infty(H^1)}||a||_V \leq \varepsilon||a||_V,$$

also given $||a||_{L^\infty(H^1)} \leq \frac{\varepsilon}{C}$.

It therefore suffices to choose D_1 such that $||A_0||_{H^1} < \varepsilon$ for some $\varepsilon > 0$ depending only on the bundle, E. We can then choose $T = T(\varepsilon, D_1, A_0) > 0$ to obtain a priori bounds of $a \in L^2(H^2) \cap C^0(H^1) \cap H^1(L^2)$. Explicitly, we have shown that

$$\Big|\Big|\Big(\frac{d}{dt} + \Delta_1\Big)a\Big|\Big|_{L^{2,2}} \leq \varepsilon||a||_V,$$

which is therefore a contraction mapping for $\varepsilon < 1$ (recall $a(0) = 0$). This implies the existence of $a \in V$, and hence $a \in L^2(H^2) \cap C^0(H^1) \cap H^1(L^2)$. Moreover a is smooth in space for $t > 0$ by the smoothing property of solutions of parabolic flows. Such a claim follows from general theory, rather than a particular theorem; however, a review of the relevant theory can be found in Chapter 4 of [18].

5.4.4 Local Existence for the Yang–Mills Flow

Now that we have proven the existence of a solution to the gauge equivalent flow (5.12), we must construct gauge transformations which transform this into a solution of the YMHF equation. Analogously to Donaldson's Ansatz, we aim to obtain a local solution to the YMHF given a solution to (5.12). This yields the gauge transformation relating D and $\bar{D}$, where $\bar{D}$ is the solution to the gauge-equivalent flow, also constructed in Section 5.4.3.

Recall that by evolving the background connection we have that

$$s = -\bar{D}^*a \in L^2(H^1)$$

and is smooth for $t > 0$. Let $\{t_k\}_{n\in\mathbb{N}}$ be a sequence such that $0 < t_k \leq T$, $t_k \searrow 0$ as $k \to \infty$. Solve

$$\begin{cases} \dfrac{dS}{dt} = S \circ s \\ S(t_k) = \text{id}, \end{cases} \tag{5.21}$$

on $[t_k, T)$ to obtain a sequence $S_k = S_k(t) \in \mathcal{G}$ of smooth gauge transformations depending smoothly on t for $0 < t \leq T$. By this smooth dependence we have $S_k = S_\ell^{-1}(t_k) \circ S_\ell$ for $\ell > k$. Then let

$$D_k = (S_k^{-1})^*\bar{D} = S_\ell(t_k)^*D_\ell$$

be the corresponding connections. For each k, $D_k = D_k(t)$ is smooth in space for $0 < t \leq T$ and is a classical solution of the YMHF evolution equation. Also note that since $\bar{D}$ is continuous in time and $S_k(t_k) = \mathrm{id}$, we get

$$D_k(t_k) = \bar{D}(t_k) \to D_0 \qquad \text{in } H^1 \tag{5.22}$$

as $k \to \infty$. This tells us that the $D_k(t_k)$ converges 'diagonally', and we would like to show that $\lim_{t\to 0} D_k(t) = D_k(0)$, and that $\lim_{k\to\infty} D_k(t) = D_0(t)$, which means that both limits exist independently of the other. This is not guaranteed by the diagonal limit existing, although by the energy inequality (5.8) and invariance of the energy under gauge transformations we have *uniform* bounds

$$\begin{aligned} \left|\left|\frac{d}{dt}D_k\right|\right|^2_{L^{2,2}} &= ||D_k^* F_{D_k}||^2_{L^{2,2}} \leq \mathcal{YM}(D_0), \\ \sup_t \mathcal{YM}\big(D_k(t)\big) &\leq \mathcal{YM}(D_0) \end{aligned} \tag{5.23}$$

for any k. This says that D_k is differentiable in time and is bounded uniformly for any time. Moreover, since time is only one dimensional, the Sobolev embedding theorem gives us that D_k is actually continuous in time. Together, this implies

$$D_k(0) = \lim_{t\searrow 0} D_k(t) \quad \text{in } L^2$$

exists. Note that we only have the limit in L^2 since we only have an L^2 bound. We would now like to show that

$$D_k(0) \to D_0 \qquad \text{in } H^1$$

as $k \to \infty$, although it turns out that this is too much to ask since we only have an L^2 bound on D_k. This guarantees the existence of a weakly convergence subsequence by the Banach-Alaoglu theorem, since L^2 is reflexive. This, together with the strong diagonal limit in H^1 guarantees

$$D_k(0) \to D_0 \qquad \text{in } L^2$$

as $k \to \infty$.

Similarly, by (5.21) we have

$$\left|\left|\frac{d}{dt}S_\ell\right|\right|^2_{L^{2,4}} \leq ||s||^2_{L^{2,4}} \leq C||s||^2_{L^2(H^1)},$$

since S is unitary. Therefore S is continuous in time and bounded for any t, which guarantees

$$S_\ell(0) = \lim_{t\searrow 0} S_\ell(t) \in L^4$$

exists for any ℓ.

Fix some $\ell = \hat{\ell}$ and let $\hat{S} = S_{\hat{\ell}}, \hat{D} = D_{\hat{\ell}}, \hat{S}_0 = \hat{S}(0), \hat{D}_0 = \hat{D}(0), \hat{S}_k = \hat{S}(t_k) \in \mathscr{G}$. In this notation, the above conclusions are expressed as

$$\begin{aligned} \hat{S}_k \to \hat{S}_0 \qquad & \text{in } L^4, \\ D_k(0) = \hat{S}_k^*(\hat{D}_0) \to D_0 \qquad & \text{in } L^2. \end{aligned}$$

Moreover, if we let $\hat{D}_0 = D_1 + \hat{A}_0, D_0 = D_1 + A_0,\ \hat{A}_0, A_0 \in L^2$, we see that

$$\begin{aligned} \hat{S}_k^*(\hat{D}_0) - D_0 &= \hat{S}_k^*(D_1) - D_1 + \hat{S}_k^{-1}\hat{A}_0\hat{S}_k - A_0 \\ &= \hat{S}_k^{-1} \circ (D_1\hat{S}_k) + \hat{S}_k^{-1}\hat{A}_0\hat{S}_k - A_0 \to 0 \qquad \text{in } L^2 \end{aligned}$$

when one considers the action on sections. Thus, also

$$\lim_{k\to\infty} D_1\hat{S}_k = \lim_{k\to\infty} (\hat{A}_0\hat{S}_k - \hat{S}_k A_0) \in L^2 \tag{5.24}$$

exists and necessarily coincides with the distributional limit $D_1\hat{S}_0$ by the uniqueness of limits; that is,

$$\hat{S}_k \to \hat{S}_0 \qquad \text{in } H^1.$$

This then implies that $D_k = \hat{S}_k^*(\hat{D})$ converges uniformly with respect to t to some

$$D = \hat{S}_0^*(\hat{D}) \in C^0(L^2)$$

with $D(0) = D_0$ and $\frac{d}{dt}D \in L^{2,2}$ by (5.23). $D \in C^0$ in time since $D \in H^1$ in time, and $D \in L^2$ since $\hat{S}_0 \in H^1$ in space, and D will have one derivative less than $\hat{S}_0$. This follows since both terms converge in C^0 in time and $\hat{S}_k$ converges in H^1 and $\hat{D}$ is in L^2, so their product is also in L^2 by the Sobolev multiplication theorems.
Similarly,

$$F_{D_k} = S_k^*\big(F(\hat{D})\big)$$

converges in L^2, locally uniformly, in that it converges uniformly away from $t = 0$, for $t > 0$. Since $D_k \to D$ in $C^0(L^2)$, we moreover have convergence

$$F_{D_k} \to F_D$$

in the distributional sense. Together, these results imply

$$F_{D_k} \to F_D \qquad \text{in } C^0((0,T];L^2)$$

by the uniqueness of limits. In the same way, from (5.23) and since $D \in C^0([0,T];L^2)$, we have by the Banach Alaoglu theorem that (possibly passing to a subsequence)

$$F_{D(t)} \rightharpoonup F_{D_0}$$

weakly in L^2 as $t \to 0$. Finally, since by (5.23) also

$$\limsup_{t\to 0} ||F_{D(t)}||^2_{L^2} \leq ||F_{D_0}||^2_{L^2},$$

we obtain strong convergence in L^2 as $t \to 0$; that is $F_D \in C^0([0,T]; L^2)$- see, for example, Proposition 3.32 of [10]. Hence D is in fact a weak solution to the Yang–Mills evolution problem YMHF. Moreover, D satisfies (5.8). This proves the first claim of Theorem 5.2.3.

5.5 Gauge Normalisation

For the proof of uniqueness we must also consider the gauge equivalent flow, (5.12), of the YMHF. This is necessary since considering solutions of equation (5.12) will yield gauge-equivalent solutions to the YMHF, and these are equally valid. Note however, that when one finds solutions equation (5.15) for $s = -D^*a$, it is necessary that the operator $D : \Omega^0(\text{ad } E) \to \Omega^1(\text{ad } E)$ be invertible so that the equation (5.21) has a unique solution. If D weren't invertible, then it is possible to have multiple $s = -D^*a$, which leads to non-unique solutions of S. Amongst other places, we have by [19] Theorem 3.1, that the operator $D : \Omega^0(\text{ad } E) \to \Omega^1(\text{ad } E)$ being injective is equivalent to the connection D being irreducible.

5.5.1 Irreducible Connections

Given a connection $D \in \Omega^1(\text{ad } E)$ on E the *isotropy subgroup* of D,

$$\Gamma = \Gamma(D) = \{S \in \mathcal{G} : S^*(D) = D\},$$

is the group of all gauge transformations which fix the connection. It is trivial to check that this is a subgroup, and if one considers the one parameter variation of an element in S, it is clear that the Lie algebra of this subgroup is

$$\gamma = \{s \in \Omega^0(\text{ad } E) : \ Ds = 0\},$$

i.e., the kernel of the exterior differential operator D. A connection is *irreducible* if and only if the isotropy subgroup contains only elements in the centre of the G, although for a proof we refer to by Lemma 4.2.8 of [15], although a proof appears in various other places.

Although to do an analysis of the flow over arbitrary base manifolds would require the analysis of reducible connections, we will follow Struwe and restrict our analysis to only be for connections which are irreducible. This does not constitute a big loss of generality, however, since if $G = SU(2)$ and D is reducible, then either E is trivial and $D = d$ is the trivial connection or E splits into a direct sum of line bundles $E = \bigoplus_{i=1}^{n} E_i$, where the connection D splits into an irreducible connection on each component, see, for example, [15]. In the first case we have $\mathcal{YM}(D) = 0$, and so there is no need for a heat flow analysis, and in the second case the analysis boils down to the case where we have an irreducible connection. It is then fortuitous that the case where $G = SU(2)$ is the case which interests us the most, and so we will only consider irreducible connections for the rest of this thesis. The centre of $SU(2)$ is trivial, and so D is irreducibile iff

$$\ker(D) \cap \Omega^0(\text{ad } E) = \{0\}.$$

For $D \in H^1$ this requirement on irreducibility is equivalent to the algebraic condition

$$||s||_{H^1} \leq C||Ds||_{L^2} \tag{5.25}$$

for $s \in H^1\big(\Omega^0(\text{ad } E)\big)$ with $C = C(D)$. This constant can be chosen locally uniformly as follows.

Lemma 5.5.1. *Suppose D_0 satisfies (5.25) with $C_0 = C(D_0)$, $D_0 \in H^1$. There exists an H^1 neighbourhood $\mathscr{V}$ of D_0 and a constant $C > 0$ such that any $D \in \mathscr{V}$ is irreducible and there holds*

$$||s||_{H^1} \leq C||Ds||_{L^2}$$

uniformly for $s \in H^1\big(\Omega^0(\text{ad } E)\big)$.

Proof. Let us suppose by contradiction that $A_k \in H^1\big(\Omega^1(\text{ad } E)\big)$, $s_k \in H^1\big(\Omega^0(\text{ad } E)\big)$ with $A_k \to 0$ in H^1 as $k \to \infty$, $||s_k||_{H^1} = 1$ for all k and that we have

$$||D_k s_k||_{L^2} = ||(D_0 + A_k)s_k||_{L^2} \to 0.$$

as $k \to \infty$. Then by Hölder's inequality and the Sobolev embedding theorem we have

$$\begin{aligned} C_0^{-1} = C_0^{-1}||s_k||_{H^1} &\leq ||D_0 s_k||_{L^2} \leq ||D_k s_k||_{L^2} + ||A_k \# s_k||_{L^2} \\ &\leq ||D_k s_k||_{L^2} + ||A_k||_{L^4}||s_k||_{L^4} \\ &\leq ||D_k s_k||_{L^2} + K||A_k||_{H^1}||s_k||_{L^4} \to 0 \end{aligned}$$

as $k \to \infty$, and so we obtain a contradiction since we assumed that C_0 was a uniform constant. □

5.5.2 Gauge Fixing

Since it is equivalent to consider the gauge-equivalent flow to minimise the Yang–Mills energy, it would be logical to find a gauge which optimises the regularity of the solution to the gauge equivalent flow, equation (5.12). Namely, we want to find a gauge which forces the flow to be *parabolic*, and, hopefully unsurprisingly, the heat flow is parabolic if and only if the Coulomb gauge condition holds. Namely, we would like to show the existence of local Coulomb gauges which depend smoothly on the connection. Let $D_0 \in H^1$ be irreducible in the sense that it satisfies (5.25) and define $D_{\text{bg}}(t) = D_1 + A_{\text{bg}}(t)$ for $0 \leq t \leq T$ be a family of background connections such that $D_{\text{bg}}(0) = D_0 = D_1 + A_0$ and A_{bg} is smooth in space for $t > 0$ and $A_{\text{bg}} \in L^2(H^2) \cap H^1(L^2)$, as determined by (5.16). Let $H^1(\mathcal{G})$ denote the H^1-closure of $\mathcal{G}$.

Proposition 5.5.2. *Let D be a weak solution of YMHF on $[0,T)$ as in Theorem 5.2.3 (i). There exists $T_0 > 0$ and a family of gauge transformations*

$$S = S(t) \in C^0\big([0,T_0]; H^1(\mathcal{G})\big)\big)$$

with

$$s = S^{-1} \circ \frac{d}{dt} S \in L^2(H^1)$$

such that

$$\bar{D} = S^*(D) = D_{bg} + \bar{a}$$

satisfies $\bar{a} \in L^\infty(H^{1,2}) \cap H^1(L^2)$, $\bar{a}(t) \to 0$ in H^1 as $t \to 0$, and $\bar{D}^*\bar{a} = 0$.

Proof. In a method analogous to constructing the solution of the gauge equivalent flow, we first consider the smooth case. We then construct sequences of smooth functions such that each term can be treated by the smooth case and then take the limit.

The

(i) Consider first the case that D_0, D_{bg} and D are smooth. For simplicity, let

$$D = D_{\text{bg}} + a =: D_a, \qquad \bar{D} = D_{\text{bg}} + \bar{a} =: D_{\bar{a}}$$

and to denote the corresponding curvatures by

$$F_{D_a} =: F_a, \qquad F_{D_{\bar{a}}} = F_{\bar{a}}.$$

Following Uhlenbeck's method of gauge constructionin in Chapter 3, we want to apply the implicit function theorem to be able to find a Coulomb gauge representative of the connection. For this, we require that $p > 2$ so that this map is smooth. For $2 < p < 4$ and a fixed $t \geq 0$ we introduce the map

$$\begin{gathered} L : W^{1,p}\big(\Omega^1(\text{ad } E)\big) \times W^{2,p}(\mathcal{G}) \to L^p\big(\Omega^0(\text{ad } E)\big) \\ L(a,S) = D^*_{\bar{a}}\bar{a}, \end{gathered}$$

where by an abuse of notation we let

$$\bar{a} = \bar{a}(a,S) = S^*(D_a) - D_{\text{bg}}.$$

Recall that for $p > 2$ we have that $W^{2,p}(\mathcal{G})$ is a Banach manifold as discussed in section 2.10. Moreover, the linearisation of this map about $S(0) = \text{id}$ is given by

$$\begin{aligned} l_{\bar{a}}(\psi) := \left|\frac{\partial L}{\partial S}\right|_{(a,S(0))} (\psi) &= \frac{d}{dt}(e^{-\psi t} D_a e^{\psi t})^*(e^{-\psi t} D_a e^{\psi t} - D_{\text{bg}})|_{t=0} \\ &= D^*_{\bar{a}} D_{\bar{a}}\psi + D_{\bar{a}}\psi \# \bar{a}, \end{aligned}$$

where $\psi \in W^{2,p}\big(\Omega^1(\text{ad } E)\big)$. This linearisation
$l_{\bar{a}} : W^{1,p}\big(\Omega^1(\text{ad } E)\big) \to L^p\big(\Omega^0(\text{ad } E)\big)$ is injective for $||\bar{a}||_{L^4}$ small enough.

To see this, multiply $l_{\bar{a}}$ with ψ to find that

$$\begin{aligned}\big(l_{\bar{a}}(\psi), \psi\big) = ||D_{\bar{a}}\psi||^2_{L^2} + (D_{\bar{a}}\psi \# \bar{a}, \psi) &\geq ||D_{\bar{a}}\psi||^2_{L^2} - ||D_{\bar{a}}\psi||_{L^2}||\bar{a}||_{L^4}||\psi||_{L^4} \\ &\geq ||D_{\bar{a}}\psi||^2_{L^2} - C||D_{\bar{a}}\psi||_{L^2}||\bar{a}||_{L^4}||\psi||_{H^1} \\ &\geq ||D_{\bar{a}}\psi||^2_{L^2} - C||D_{\bar{a}}\psi||^2_{L^2}||\bar{a}||_{L^4} \geq 0\end{aligned}$$

for $||\bar{a}||_{L^4}$ sufficiently small where $C = C(M, D)$ depends on the Sobolev constant and the uniform constant as determined by Lemma 5.5.1. Therefore the operator $l_{\bar{a}}$ is invertible if $t \geq 0$ and $||\bar{a}||_{L^4}$ is sufficiently small. It follows by the implicit function theorem that there exists $T_0 > 0$ and $S = S(t) \in C^1\big([0, T_0]; W^{2,p}(\mathcal{G})\big)$ such that $S(0) = \mathrm{id}$ and

$$L(a(t), S(t)) = 0;$$

that is, $S^*(D_a) = D_{\bar{a}}$ is of class $C^1(W^{1,p})$ and satisfies $D^*_{\bar{a}}\bar{a} = 0$.

(ii) Since the above construction is only for smooth connections and the connection D in the hypothesis is in fact only a weak solution of the YMHF, we adopt a method analogous to the one we used to construct a weak solution of the YMHF. Namely, we construct sequences of smooth solutions to the YMHF and find a gauge transformation to put each of these terms in Coulomb gauge and then take a limit, as follows: Firstly, let

$$D_k = \hat{S}^*_k(\hat{D}) = D_{\mathrm{bg}} + a_k,$$

on $[t_k, T)$, where $\hat{D}$ is the same as in the construction of the weak solution to the YMHF. The existence of such a sequence is guaranteed by the construction of the weak solution to the YMHF in the local existence Section 5.4. As before, we have the initial condition

$$D_k(t_k) = D_{k_0} \to D_0 \text{ in } H^1$$

for some sequence $t_k \to 0$ as $k \to \infty$. Although such notation seems somewhat circuitous, it is necessary for our argument. We can understand D_{k_0} to mean the initial connection for $t \in (t_k, T_k]$. Note that

$$\left|\frac{d}{dt}D_k\right| = \left|\frac{d}{dt}D\right| \in L^{2,2},$$

by construction (recall equation (5.23)). We must also choose corresponding smooth background connections $D_{\mathrm{bg},k} = D_1 + A_{\mathrm{bg},k}$, where $A_{\mathrm{bg},k}$ solves (5.16) with initial data

$$A_{\mathrm{bg},k}(t_k) = A_{\mathrm{bg}}(t_k) + a_k(t_k).$$

Note that by the above definitions we have

$$D_k(t_k) = D_{\mathrm{bg},k}(t_k). \tag{5.26}$$

Also note that by Lemma (5.5.1) the data D_{k_0} satisfy condition (5.25) with a uniform constant C. Moreover, $A_{\mathrm{bg},k} \in V_T$, and, given $\varepsilon > 0$, by a suitable choice of D_1 and choosing a smaller time $T > 0$ if necessary, we can achieve that

$$\varepsilon_k(T) := \left|\left|\frac{d}{dt}D_k\right|\right|^2_{L^{2,2}} + \left|\left|\frac{d}{dt}A_{\text{bg},k}\right|\right|^2_{L^{2,2}} + ||A_{\text{bg},k}||^2_{L^\infty(H^1)} + ||A_{\text{bg},k}||^2_{L^2(H^2)} + ||F_{\text{bg},k}||^2_{L^2(H^1)} < \varepsilon,$$

uniformly in k, $k \geq k_0(\varepsilon)$. This control comes from the fact that each term except for $||A_{\text{bg},k}||^2_{L^\infty(H^1)}$ can be made arbitrarily small by making the time interval small enough, again by the absolute continuity of the Lebesgue integral. For $||A_{\text{bg},k}||^2_{L^\infty(H^1)}$ this method is invalid, since decreasing the time interval does not necessarily decrease the L^∞ norm. Fortunately, by (5.17) we may make $||A_{\text{bg},k}||^2_{L^\infty(H^1)}$ as small as needed by making A_0 as small as needed.

Since (5.26) holds we may apply the construction used in part (i) to yield a C^1(in time)-family of smooth gauge transformations $S_k = S_k(t)$ on some interval $[t_k, T_k]$ for $t_k < T_k \leq T$, such that

$$\bar{D}_k = S_k^*(D_k) = D_{\text{bg},k} + \bar{a}_k$$

satisfies

$$\bar{D}_k^* \bar{a}_k = 0. \tag{5.27}$$

Since we aim to apply the implicit function theorem argument to each term in the sequence $\bar{a}_k$, for each k there exists a corresponding time, T_k, which is the maximal time for which the implicit function theorem holds. Since we apply this argument to *every* element in the sequence for k large enough, the infimum of T_k over all k will characterise the maximum existence time of the flow. Therefore this term T_k is crucial in our argument, and if it were to be 0 then our argument would break down. The goal of the following lemma is two-fold: Firstly, we need to establish the existence of a $T_0 > 0$ such that $T_k \geq T_0$ for each k large enough. Secondly, we need to obtain suitable a-priori bounds on the terms $\bar{a}_k$ and S_k so that we may pass to the limit $k \to \infty$. Since the background connection is evolving we have that $||\bar{a}_k||_{L^{\infty,4}} \leq \varepsilon$, and this implies that there exists a corresponding $T_k > 0$ for each k large enough. Unfortunately, this doesn't yield the existence of a $T_0 > 0$ such that each $T_k \geq T_0$ for each k large enough, since simply taking the infimum over all the T_k may in fact yield 0.

To be able to extend the definition of $\bar{a}_k$ to $[0, T_k]$, we let $s_k = S_k^{-1} \circ \frac{d}{dt} S_k$ and extend $s_k(t) = 0$, $\bar{a}_k(t) = 0$, that is $S_k(t) = \text{id}$, $D_{\text{bg},k}(t) = D_k(t) = D_{k_0}$ for $0 \leq t \leq t_k$.

Lemma 5.5.3. *There exists constants $C = C(D_0),\ T_0 = T(D_0)$ such that for any $0 < T \leq T_0$ and sufficiently large k there holds*

$$\left|\left|\frac{d}{dt}\bar{a}_k\right|\right|^2_{L^{2,2}} + ||\bar{a}_k||^2_{L^\infty(H^1)} + ||s_k||^2_{L^2(H^1)} \leq C\varepsilon_k(T)$$

Proof. The proof requires several steps. Since this argument applies to each term of the sequence we will drop the reference k for ease of notation.

Claim 5.5.4. *There exist constants $C = C(D_0) > 0$, $T_0 = T(D_0) > 0$, $\varepsilon > 0$ such that for sufficiently large k we have*

$$\Big|\Big|\frac{d}{dt}\bar{a}\Big|\Big|^2_{L^{2,2}} + ||s||_{L^2(H^1)} \leq C\varepsilon(T)$$

for $0 < T \leq T_0$, provided $||\bar{a}||_{L^{\infty,4}} \leq \varepsilon$.

Proof. Firstly we must differentiate (5.27). Note that for $\sigma \in \Omega^0(E)$ we have

$$\begin{aligned}\frac{d}{dt}D_{\bar{a}}\sigma &= \frac{d}{dt}\big(S^{-1}D_a S\big)\sigma \\ &= -sD_{\bar{a}}\sigma + S^*\Big(\frac{d}{dt}D_a\Big)\sigma + S^{-1}D_a\Big(\frac{d}{dt}S\sigma\Big) \\ &= S^*\Big(\frac{d}{dt}D_a\Big)\sigma + (D_{\bar{a}}s)\sigma.\end{aligned}$$

Therefore, since $\bar{a} = D_{\bar{a}} - D_{\text{bg}}$, we find that

$$\frac{d}{dt}\bar{a} = D_{\bar{a}}s + S^*\Big(\frac{d}{dt}D_a\Big) - \frac{d}{dt}D_{\text{bg}} = D_{\bar{a}}s + f,$$

where

$$f = \Big(S^*\Big(\frac{d}{dt}D_a\Big) - \frac{d}{dt}D_{\text{bg}}\Big).$$

We have then that

$$0 = \frac{d}{dt}\big(D^*_{\bar{a}}\bar{a}\big) = D^*_{\bar{a}}D_{\bar{a}}s + D_{\bar{a}}s\#\bar{a} + D^*_{\bar{a}}f + f\#\bar{a}.$$

If we multiply by s and integrate in time, after multiple applications of Hölder's inequality we find that

$$\begin{aligned}||D_{\bar{a}}s||^2_{L^{2,2}} &= (D^*_{\bar{a}}D_{\bar{a}}s, s) \\ &\leq ||D_{\bar{a}}s||_{L^{2,2}}(||\bar{a}||_{L^{\infty,4}}||s||_{L^{2,4}} + ||f||_{L^{2,2}}) + ||f||_{L^{2,2}}||\bar{a}||_{L^{\infty,4}}||s||_{L^{2,4}},\end{aligned}$$

Since

$$||f||_{L^{2,2}} \leq \Big|\Big|\frac{d}{dt}D_a\Big|\Big|_{L^{2,2}} + \Big|\Big|\frac{d}{dt}D_{\text{bg}}\Big|\Big|_{L^{2,2}} \leq C\varepsilon(T)$$

by Minkowski's inequality and the linearity of the Lebesgue integral. Morevoer, by the irreducibility condition, equation (5.25), and the Sobolev embedding we also have

$$||s||_{L^{2,4}} \leq C||s||_{L^2(H^1)} \leq C||D_{\bar{a}}s||_{L^{2,2}},$$

where C depends on the locally uniform constant as in Lemma 5.5.1 and the Sobolev embedding constant. Therefore, we find that

$$||D_{\bar{a}}s||_{L^{2,2}} \leq C(||\bar{a}||_{L^{\infty,4}}||s||_{L^{2,4}} + ||f||_{L^{2,2}} + ||f||_{L^{2,2}}||\bar{a}||_{L^{\infty,4}}),$$

which yields the desired estimate for s. Finally, since

$$\frac{d}{dt}\bar{a} = \frac{d}{dt}(S^*(D_a) - D_{\text{bg}}) = D_{\bar{a}}s + f,$$

the claim follows by taking T such that f is small enough. $\square$

Let $\delta > 0$ be as in Lemma (5.3.3) and let R be chosen such that (5.11) is satisfied, which is possible because $F_D \in C^0(L^2)$. By (5.8) we may choose $R = R(D_0)$ on a time interval of sength CR^2.

Claim 5.5.5. *There exist constants $C = C(D_0)$, $T_0 = T(D_0) > 0$, $\varepsilon = \varepsilon(D_0) > 0$ such that*

$$||\bar{a}||^2_{L^{\infty,4}} \leq C\big(||F_{\bar{a}} - F_{bg}||^2_{L^{\infty,4}} + \varepsilon(T)\big)$$

for $0 < T \leq T_0$, provided $||\bar{a}||_{L^{\infty,4}} \leq \varepsilon$.

Proof. By Lemma (5.3.3) for any t we have

$$||\bar{a}||^2_{L^4} \leq C||D_{\bar{a}}\bar{a}||^2_{L^2} + CR^{-2}||\bar{a}||^2_{L^2},$$

since $\bar{a}$ is in Coulomb gauge. By the definition of curvature there holds

$$D_{\bar{a}}\bar{a} = F_{\bar{a}} - F_{\text{bg}} + \bar{a}\#\bar{a}.$$

Note that by Jensen's inequality we have

$$\frac{d}{dt}||\bar{a}||^2_{L^2} \leq \left|\left|\frac{d}{dt}\bar{a}\right|\right|^2_{L^2}.$$

By integrating in time and since $\bar{a}(0) = 0$, together with Claim 5.5.4 we have

$$||\bar{a}||^2_{L^{\infty,2}} \leq \left|\left|\frac{d}{dt}\bar{a}\right|\right|^2_{L^{2,2}} \leq C\varepsilon(T).$$

Together, this yields

$$||\bar{a}||^2_{L^{\infty,4}} \leq C||F_{\bar{a}} - F_{\text{bg}}||^2_{L^{\infty,2}} + C_1||\bar{a}||^4_{L^{\infty,4}} + C_2TR^2\varepsilon(T),$$

and the claim follows. $\square$

Claim 5.5.6. *Under the assumptions of Claim 5.5.5 there holds*

$$||\bar{a}||_{L^\infty(H^1)} \leq C\Big(||F_{\bar{a}} - F_{bg}||^2_{L^{\infty,2}} + \varepsilon(T)\Big)$$

Proof. By definition, we have $D_{\bar{a}}\bar{a} = D_{\text{bg}}\bar{a} + (A_1 + A_{\text{bg}} + \bar{a})\#\bar{a}$, and so for fixed t it suffices to estimate

$$||\nabla_{\text{ref}}\bar{a}||_{L^2} \leq ||\bar{\nabla}\bar{a}||_{L^2} + ||(A_1 + A_{bg} + \bar{a})\#\bar{a}||_{L^2},$$

where $\bar{\nabla}$ is the covariant derivative corresponding to $\bar{D} = \bar{D}_{\bar{a}}$. Now, by Lemma 5.3.3, (5.11), (5.27) and the first variation of the curvature we have

$$\begin{aligned} ||\bar{\nabla}\bar{a}||_{L^2}^2 &\leq C||\bar{D}\bar{a}||_{L^2}^2 + CR_0^2||\bar{a}||_{L^2}^2 \\ &\leq C||F_{\bar{a}} - F_{\text{bg}}||_{L^2}^2 + C||\bar{a}||_{L^4}^4 + CTR^{-2}\varepsilon(T), \end{aligned}$$

and in view of Claim 5.5.5 the assertion follows. □

Claim 5.5.7. *Under the assumptions of Claim 5.5.5 above there holds*

$$||F_{\bar{a}} - F_{bg}||_{L^{\infty,2}}^2 \leq C\varepsilon(T).$$

Proof. By (5.12) and the second Bianchi identity there holds

$$\frac{d}{dt}F_{\bar{a}} + \Delta_{\bar{a}}F_{\bar{a}} = D_{\bar{a}}D_{\bar{a}}s,$$

since for any $\sigma \in \Omega^0(\text{ad } E)$, we have

$$\begin{aligned} \frac{d}{dt}(D_{\bar{a}} \circ D_{\bar{a}})\sigma &= \left(\frac{d}{dt}D_{\bar{a}} \circ D_{\bar{a}})\sigma + (D_{\bar{a}} \circ \frac{d}{dt}D_{\bar{a}}\right) \\ &= (-D_{\bar{a}}^*F_{\bar{a}} + D_{\bar{a}}s)D_{\bar{a}}\sigma + D_{\bar{a}}(-D_{\bar{a}}^*F_{\bar{a}}\sigma + D_{\bar{a}}s\sigma) \\ &= (-D_{\bar{a}}^*F_{\bar{a}} + D_{\bar{a}}s)D_{\bar{a}}\sigma - D_{\bar{a}}D_{\bar{a}}^*F_{\bar{a}}\sigma \\ &\qquad\qquad + D_{\bar{a}}D_{\bar{a}}s\sigma + D_{\bar{a}}^*F_{\bar{a}}D_{\bar{a}}\sigma - D_{\bar{a}}sD_{\bar{a}}\sigma \\ &= -D_{\bar{a}}D_{\bar{a}}^*F_{\bar{a}}\sigma + D_{\bar{a}}D_{\bar{a}}s\sigma \\ \implies \frac{d}{dt}F_{\bar{a}} &= D_{\bar{a}}D_{\bar{a}}s - D_{\bar{a}}D_{\bar{a}}^*F_{\bar{a}} = D_{\bar{a}}\big(\frac{d}{dt}D_{\bar{a}}\big). \end{aligned}$$

We also have

$$\frac{d}{dt}(F_{\bar{a}} - F_{\text{bg}}) + \Delta_{\bar{a}}(F_{\bar{a}} - F_{\text{bg}}) = D_{\bar{a}}D_{\bar{a}}s - \frac{d}{dt}F_{\text{bg}} - \Delta_{\bar{a}}F_{\text{bg}}.$$

If we multiply by $F_{\bar{a}} - F_{\text{bg}}$ and integrate in time, we get

$$\begin{aligned} \text{I} := \frac{1}{2}||F_{\bar{a}} - F_{\text{bg}}||_{L^{\infty,2}} + ||D_{\bar{a}}(F_{\bar{a}} - F_{\text{bg}})||_{L^{2,2}}^2 + ||D_{\bar{a}}^*(F_{\bar{a}} - F_{\text{bg}})||_{L^{2,2}}^2 \\ \leq \text{II+III+IV}. \end{aligned}$$

By the second Bianchi identity, Claim 5.5.4, Hölder's inequality and the assumption that $||\bar{a}||_{L^{\infty,4}} \leq \varepsilon$, we get

$$\begin{aligned} \text{II} &= (D_{\bar{a}}D_{\bar{a}}s, F_{\bar{a}}) - (D_{\bar{a}}D_{\bar{a}}s, F_{\text{bg}}) = \cancelto{0}{(s, D_{\bar{a}}^*D_{\bar{a}}^*F_{\bar{a}})} - (D_{\bar{a}}s, D_{\bar{a}}^*F_{\text{bg}}) = -(D_{\bar{a}}s, D_{\bar{a}}^*F_{\text{bg}}) \\ &= -(D_{\bar{a}}s, D_{\text{bg}}^*F_{\text{bg}} + \bar{a}\#F_{\text{bg}}) \\ &\leq ||D_{\bar{a}}s||_{L^{2,2}}\big(||F_{\text{bg}}||_{L^2(H^{1,2})} + ||\bar{a}||_{L^{\infty,4}}||F_{\text{bg}}||_{L^{2,4}}\big) \leq C\varepsilon(T) \end{aligned}$$

To estimate III we see that

$$\begin{aligned}
\mathrm{III} &= \Big(\frac{d}{dt}F_{\mathrm{bg}}, F_{\bar a} - F_{\mathrm{bg}}\Big) = \Big(D_{\mathrm{bg}}\Big(\frac{d}{dt}D_{\mathrm{bg}}\Big), F_{\bar a} - F_{\mathrm{bg}}\Big) \\
&= \Big(\frac{d}{dt}D_{\mathrm{bg}}, D^*_{\mathrm{bg}}(F_{\bar a} - F_{\mathrm{bg}})\Big) \\
&= \Big(\frac{d}{dt}D_{\mathrm{bg}}, D^*_{\bar a}(F_{\bar a} - F_{\mathrm{bg}}) + \bar a \# (F_{\bar a} - F_{\mathrm{bg}})\Big) \\
&\le C\varepsilon(T) + \frac{1}{4}||D^*_{\bar a}(F_{\bar a} - F_{\mathrm{bg}})||^2_{L^{2,2}} + C||\bar a||^2_{L^{\infty,4}}||F_{\bar a} - F_{\mathrm{bg}}||^2_{L^{2,4}},
\end{aligned}$$

by Hölder and Young's inequality. Since $\left|\frac{d}{dt}D_{\mathrm{bg}}\right| \in L^{2,2}$ we may shrink the time interval by the absolute continuity of the Lebesgue integral such that $\left|\left|\frac{d}{dt}D_{\mathrm{bg}}\right|\right|^2_{L^{2,2}} \le \varepsilon(T)$. We need to check that $||F_{\bar a} - F_{\mathrm{bg}}||^2_{L^{2,4}}$ is bounded so that we can make $||\bar a||_{L^{\infty,4}}$ small enough to make the whole second term $\le C\varepsilon(T)$. By Lemma 5.3.3,

$$\begin{aligned}
||F_{\bar a} - F_{\mathrm{bg}}||^2_{L^{2,4}} \le C\big(||D^*_{\bar a}(F_{\bar a} - F_{\mathrm{bg}})||^2_{L^{2,2}} + ||D_{\bar a}(F_{\bar a} - F_{\mathrm{bg}})||^2_{L^{2,2}}\big) \\
+ CTR^{-2}||F_{\bar a} - F_{\mathrm{bg}}||_{L^{\infty,2}},
\end{aligned}$$

which yields for small $\varepsilon > 0, T > 0$

$$\mathrm{III} \le C\varepsilon(T) + \frac{1}{3}\mathrm{I}.$$

Lastly, by Hölder and Young's inequalities we estimate

$$\begin{aligned}
\mathrm{IV} &= (-\Delta_{\bar a}F_{\mathrm{bg}}, F_{\bar a} - F_{\mathrm{bg}}) \\
&= -(D_{\bar a}F_{\mathrm{bg}}, D_{\bar a}(F_{\bar a} - F_{\mathrm{bg}})) - (D^*_{\bar a}F_{\mathrm{bg}}, D^*_{\bar a}(F_{\bar a} - F_{\mathrm{bg}})) \\
&\le C\Big(||D_{\bar a}F_{\mathrm{bg}}||^2_{L^2,2} + ||D^*_{\bar a}F_{\mathrm{bg}}||^2_{L^{2,2}}\Big) + \frac{1}{3}I \\
&\le C\Big(||F_{\mathrm{bg}}||^2_{L^2(H^{1,2})} + ||\bar a||_{L^{\infty,4}}||F_{\mathrm{bg}}||^2_{L^{2,4}}\Big) + \frac{1}{3}I \\
&\le C\varepsilon(T) + \frac{1}{3}I.
\end{aligned}$$

Therefore $I \le C\varepsilon(T)$, as desired. □

In view of the above claims, there exists $T_0, C > 0$ such that

$$||\bar a_k||^2_{L^2(H^1)} \le C\varepsilon(T) < \varepsilon$$

on $[0, T]$ for any $T \le T_0$ and sufficiently large $k \ge k_0(\varepsilon)$, where ε is the constant as in the above claims.

The assertion of the lemma follows. □

Proof of 5.5.2 continued.

By Lemma 5.5.3, we can choose $0 < T_0 \le T$ such that for large k the linearised operator $l_{\bar a_k}$ corresponding to the gauge condition (5.27) are uniformly invertible on $[0, T_0]$. Therefore $T_k \ge T_0$ for large enough k. Moreover, Lemma 5.5.3 implies that the sequence $\bar a_k$ is uniformly equicontinuous in L^2, where we take $\delta = C\varepsilon(T)$ in this case. Since each $\bar a_k \in H^1$, we have by Rellich's theorem (see, for instance, Chapter 3.2.3 of [22]) that the sequence is also pointwise relatively compact in L^2. The Arzelà-Ascoli Theorem (see, for instance, Appendix C.7 of [17]) then gives the uniform convergence of a sub-sequence

$$\bar{a}_k \to \bar{a} \text{ in } C^0(L^2).$$

Hence, since $D_{\text{bg},k} \to D_{\text{bg}}$ by construction, we have

$$\bar{D}_\ell = D_{\text{bg},k} + \bar{a}_k \to \bar{D} = D_{\text{bg}} + \bar{a}$$

uniformly in L^2 as $k \to \infty$. Passing to the limit in (5.27), we obtain

$$D^*_{\bar{a}}\bar{a} = 0.$$

Finally, $\frac{d}{dt}\bar{a} \in L^{2,2}$ and since $||\bar{a}_k||_{H^1}$ os bounded, we have $\bar{a} \in L^\infty(H^1)$ by lower semi-continuity and

$$||\bar{a}(t)||_{H^1} \leq C\varepsilon(t) \to 0$$

as $t \to 0$.

Similarly, $S_k \to S$ uniformly in L^2 with $s = S^{-1}\dfrac{d}{dt}S \in L^2(H^1)$. In an almost identical way to (5.24), one can show that the limit exists uniformly and calculate the distributional limit to infer that $S_k \to S$ uniformly in H^1, and

$$D_{\bar{a}} = S^*(D_a).$$

This concludes the proof. □

5.6 Uniqueness

Now that we have shown the existence of a solution to the YMHF problem, it is now a natural step to try and show uniqueness. It should be clear that without the gauge fixing condition constructed in the previous section that a solution of the Yang–Mills equations is highly non-unique, in that every gauge equivalent connection is also a solution. In [30], Theorem 6.1, Kozono, Maeda and Naioto show uniqueness *modulo gauge transforms*. In their proof they do not require that D be irreducible, although the regularity assumption is stronger. In this section we continue to follow Struwe's method and firstly prove that every point in time there exists a gauge transform which brings the connection into Coulomb gauge, and then we show that this solution is unique.

Given $D_0 \in H^1$, a family of background connections D_{bg} as in Section 5.4, let $D_a = D_{\text{bg}} + a$ be a local weak solution to YMHF and $D_{\bar{a}} = D_{\text{bg}} + \bar{a}$ the corresponding family of normalised connections according to Proposition 5.5.2. By construction we have that $D_{\bar{a}}$ weakly solves the initial value problem

$$\frac{d}{dt}D_{\bar a} = -D^*_{\bar a}F_{\bar a} + D_{\bar a}s, \tag{5.28}$$

$$D^*_{\bar a}(\bar a) = 0, \tag{5.29}$$

$$\bar a(0) = 0, \tag{5.30}$$

where $F_{\bar a} = F_{D_{\bar a}}$, and

$$\begin{aligned} \bar a \in L^\infty\Big([0,T]; H^1\big(\Omega^1(\text{ad } E)\big)\Big) \cap H^1\Big([0,T]; L^2\big(\Omega^1(\text{ad } E)\big)\Big), \\ F_{\bar a} \in C^0\Big([0,T] : L^2\big(\Omega^2(\text{ad } E)\big)\Big), \\ s \in L^2\Big([0,T]; H^1\big(\Omega^0(\text{ad } E)\big)\Big), \end{aligned} \tag{5.31}$$

on some interval $[0,T]$, as determined by T_0 in the previous section. $\bar a$ attains its initial data in the H^1-sense. The following result shows - provided D_0 is irreducible - the solution $D_{\bar a}$ is unique.

Proposition 5.6.1. *For any $D_0 \in H^1$ satisfying the irreducibility criterion, equation 5.25, there exists $T > 0$ and a unique solution $(\bar a, s)$ of (5.28)-(5.30) on $[0,T]$ satisfying (5.31). In addition, $\bar a \in L^2(H^2)$, and $\bar a$ and s are smooth for $t > 0$. Finally, if D_0 is smooth, $\bar a$ and s are smooth for all $t \in [0,T]$.*

Proof. (i) In the previous section we showed the existence of a solution $(\bar a, s)$.

(ii) To see higher (spacial) regularity and uniqueness, we must firstly establish suitable a-priori estimates for solutions in the above class.

Estimate for s. From the second Bianchi identity and by applying $D^*_{\bar a}$ to (5.28), we have

$$D^*_{\bar a}D_{\bar a}s = D^*_{\bar a}\Big(\frac{d}{dt}D_{\bar a}\Big) = D^*_{\bar a}\Big(\frac{d}{dt}A_{\text{bg}}\Big) + \bar a \# \frac{d}{dt}\bar a.$$

The last equality comes from the expanding $D_{\bar a} = D_{\text{bg}} + \bar a$ in equation (5.28) and then differentiating. Note that the extra term is absorbed by the arbitrariness of $\#$. Multiplying by s and integrating in time, we find by an application of Hölder's inequality and Sobolev embedding theorem that

$$\begin{aligned} ||D_{\bar a}s||^2_{L^{2,2}} &= (D^*_{\bar a}D_{\bar a}s, s) = \Big(\frac{d}{dt}A_{\text{bg}}, D_{\bar a}s\Big) + \Big(\bar a \# \frac{d}{dt}\bar a, s\Big) \\ &\leq C\Big|\Big|\frac{d}{dt}A_{\text{bg}}\Big|\Big|_{L^{2,2}}||D_{\bar a}s||_{L^{2,2}} + C||\bar a||_{L^\infty(H^1)}\Big|\Big|\frac{d}{dt}\bar a\Big|\Big|_{L^{2,2}}||s||_{L^2(H^1)}, \end{aligned}$$

and by the irreducibility criterion of the connection and Young's inequality,

$$||s||^2_{L^2(H^1)} \leq C||D_{\bar a}s||^2_{L^{2,2}} \leq C\Big|\Big|\frac{d}{dt}A_{\text{bg}}\Big|\Big|^2_{L^{2,2}} + C||\bar a||^2_{L^\infty(H^1)}\Big|\Big|\frac{d}{dt}\bar a\Big|\Big|^2_{L^{2,2}}.$$

This establishes the stated regularity for s.

Estimate for $\bar{a}$: By using (5.29) we may rewrite (5.28) as

$$\frac{d}{dt}\bar{a} + D_{\bar{a}}^* F_{\bar{a}} + D_{\bar{a}} D_{\bar{a}}^* \bar{a} = D_{\bar{a}} s - \frac{d}{dt} A_{\text{bg}}.$$

Since $F_{\bar{a}} = F_{\text{bg}} + D_{\text{bg}}\bar{a} + \bar{a}\#\bar{a}$ we may also write $F_{\bar{a}} = F_{\text{bg}} + D_{\bar{a}}\bar{a} + \bar{a}\#\bar{a}$, where the extra term is absorbed into the arbitrary multilinear map. Therefore, the left hand side equals

$$\Big(\frac{d}{dt} + \Delta_{\bar{a}}\Big)\bar{a} + D_{\bar{a}}^* F_{\text{bg}} + D_{\bar{a}}^*(\bar{a}\#\bar{a}).$$

By expanding out $\Delta_{\bar{a}}$ analogously to (5.3), we see that the above expression differs from $\left(\frac{d}{dt} + \Delta_1\right)\bar{a}$ by the error terms

$$\begin{aligned}\nabla_1 \bar{a}\#\bar{a} + \bar{a}\#\bar{a}\#\bar{a} + A_{\text{bg}}\#\nabla_1\bar{a} + \nabla_1 A_{\text{bg}}\#\bar{a} + A_{\text{bg}}\#\bar{a}\#\bar{a}& \\ + A_{\text{bg}}\#A_{\text{bg}}\#\bar{a} + D_{\bar{a}}^* F_{\text{bg}}&.\end{aligned}$$

Therefore, if $||\bar{a}||_{L^\infty(H^1)}$ is sufficiently small, which is possible by Lemma 5.5.3, from Lemma 5.3.2 we estimate (by bringing the terms we don't want to the other side and throwing them away because they are negative)

$$||\bar{a}||_V^2 = \Big|\Big|\frac{d}{dt}\bar{a}\Big|\Big|_{L^2}^2 + ||\bar{a}||_{L^2(H^2)} \leq C\Big(||D_{\bar{a}} s||_{L^{2,2}}^2 + ||A_{\text{bg}}||_V^2\Big) \leq C||A_{\text{bg}}||_V^2.$$

This establishes the stated regularity of $\bar{a}$. The smoothness for $\bar{a}$ $t > 0$ by the smoothing property of the heat equation. Then, since $\bar{a}$ is smooth for $t > 0$ and $s = -D_{\bar{a}}^* \bar{a}$, we find that s is also smooth for $t > 0$, again by the smoothing effect of parabolic PDE.

Remark 5.6.2. Although showing uniqueness of the solution without gauge fixing would be impossible, our choice of gauge was not arbitrary. As in Chapter 3, where choosing the Coulomb gauge makes the functional elliptic, it is choosing in the Coulomb gauge that we make the evolution equation *parabolic*.

(iii) Next we derive similar estimates for the difference (α, σ) of two solutions $(\bar{a}_1, s_1)$, $(\bar{a}_2, s_2)$ of (5.28), (5.29) with $\bar{a}_1(0) = \bar{a}_2(0) = \bar{a}(0) = 0$. Let $D_1 = D_{\bar{a}_1}$ and so on. Note that (α, σ) satisfies

$$\frac{d}{dt}\alpha = -\big(D_1^* F_1 - D_2^* F_2\big) + D_1 s_1 - D_2 s_2$$

by (5.28). We also have

$$D_1 s_1 - D_2 s_2 = D_{\bar{a}}\sigma + \bar{a}\#\sigma + \alpha\# s, \tag{5.32}$$

denoting by $\bar{a}$ any convex linear combinations of $\bar{a}_1$ and $\bar{a}_2$, and similarly for s. To see this, let $D_1 = D_{\bar{a}} + \gamma\alpha$, $D_2 = D_{\bar{a}} - \beta\alpha$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. From this, we see that

$$\begin{aligned}D_1 s_1 - D_2 s_2 &= D_{\bar{a}}\sigma + \gamma(a_1 - a_2)s_1 + \beta(a_1 - a_2)s_2 \\ &= D_{\bar{a}}\sigma + \gamma(a_1 - a_2)s_1 + \beta(a_1 - a_2)s_2 \\ &\qquad + (\beta a_1 + \gamma a_2)(s_1 - s_2) - (\beta a_1 + \gamma a_2)(s_1 - s_2).\end{aligned}$$

We can simply rearrange this to achieve the desired result. Furthermore, note that for $i = 1, 2$, we have

$$D_i^* F_i = D_{\bar a}^* F_i + \alpha \# F_i.$$

Note also that for $\sigma \in \Omega^0(\text{ad } E)$, we have

$$\begin{aligned}
(F_1 - F_2)\sigma &= (D_{\bar a} + \beta\alpha)(D_{\bar a}\sigma + \gamma\alpha\sigma) - (D_{\bar a} - \gamma\alpha)(D_{\bar a}\sigma - \beta\alpha\sigma) \\
&= D_{\bar a}D_{\bar a}\sigma + \gamma(D_{\bar a}\alpha)\sigma - \\
&\qquad \gamma\alpha D_{\bar a}\sigma + \beta\alpha D_{\bar a}\sigma + \gamma\beta\alpha(\alpha\sigma) - D_{\bar a}D_{\bar a}\sigma + \beta(D_{\bar a}\alpha)\sigma \\
&\quad - \beta D_{\bar a}\sigma + \gamma\alpha D_{\bar a}\sigma - \beta\gamma\alpha(\alpha\sigma) = (D_{\bar a}\alpha)\sigma \\
\implies \quad F_1 - F_2 &= D_{\bar a}\alpha \\
&= D_{\bar a} + \alpha(\bar a) - \alpha(\bar a) \\
&= D_{\bar a}\alpha + \alpha \# \bar a,
\end{aligned}$$

and also

$$D_{\bar a}^*\alpha = D_{\bar a}^* a_1 - D_{\bar a}^* a_2 = \alpha \# \bar a.$$

Estimate for σ: By the second Bianchi identity and the previous estimates, we have

$$\begin{aligned}
D_{\bar a}^* D_{\bar a}\sigma &= D_{\bar a}^*\Big(\frac{d}{dt}\alpha\Big) + D_1^* D_1^* F_1 - D_2^* D_2^* F_2 + \alpha \# D_{\bar a}^* F_{\bar a} + D_{\bar a}^*(\bar a \# \sigma_\alpha \# s) \\
&= \alpha \# \frac{d}{dt}\bar a + \frac{d}{dt}\alpha \# \bar a + \alpha \# D_{\bar a}^* F_{\bar a} + D_{\bar a}^*(\bar a \# \sigma_\alpha \# s).
\end{aligned}$$

Note that we may achieve such combinations by adding and subtracting terms simultaneously from the equation, and then the minus signs disappear when an arbitrary multilinear map is considered. Multiplying by σ, integrating by parts in the last term, and from (5.25), we obtain

$$\begin{aligned}
&||\sigma||^2_{L^2(H^1)} \leq C||D_{\bar a}\sigma||^2_{L^{2,2}} = C(D_{\bar a}^* D_{\bar a}\sigma, \sigma) \\
&\leq C\bigg(\bigg|\bigg|\frac{d}{dt}\bar a\bigg|\bigg|_{L^{2,2}} ||\alpha||_{L^{\infty,4}} + ||\bar a||_{L^{\infty,4}}\bigg|\bigg|\frac{d}{dt}\alpha\bigg|\bigg|_{L^{2,2}} + ||D_{\bar a}^* F_{\bar a}||_{L^{2,2}}||\alpha||_{L^{\infty,4}}\bigg)||\sigma||_{L^{2,4}} \\
&\qquad\qquad + C\big(||\bar a||_{L^{\infty,4}}||\sigma||_{L^{2,4}} + ||\alpha||_{L^{\infty,4}}||s||_{L^{2,4}}\big)||D_{\bar a}\sigma||_{L^{2,2}}.
\end{aligned}$$

Therefore, by the irreducibility of the connection and the Sobolev embedding, we get

$$\begin{aligned}
||\sigma||^2_{L^2(H^1)} \leq C||D_{\bar a}\sigma||^2_{L^{2,2}} &\leq C\bigg(\bigg|\bigg|\frac{d}{dt}\bar a\bigg|\bigg|^2_{L^{2,2}} + ||D_{\bar a}^* F_{\bar a}||^2_{L^{2,2}} + ||s||^2_{L^2(H^1)}\bigg)||\alpha||^2_{L^\infty(H^1)} \\
&\quad + ||\bar a||^2_{L^\infty(H^1)}\bigg|\bigg|\frac{d}{dt}\alpha\bigg|\bigg|^2_{L^{2,2}} + C||\bar a||^2_{L^\infty(H^1)}||\sigma||^2_{L^2(H^1)}.
\end{aligned}$$

Therefore, for $T > 0$ small enough, we see that

$$||\sigma||^2_{L^2(H^1)} \leq C\varepsilon\bigg(\bigg|\bigg|\frac{d}{dt}\alpha\bigg|\bigg|^2_{L^{2,2}} + ||\alpha||^2_{L^\infty(H^1)}\bigg) \longrightarrow 0$$

as $T \to 0$.

Estimate for α: From the equations for α, we see that

$$\frac{d}{dt}\alpha + \Delta_{\bar{a}} = \alpha \# F_{\bar{a}} + D^*_{\bar{a}}(\alpha \# \bar{a}) + D_{\bar{a}}(\alpha \# \bar{a}) + D_{\bar{a}}\sigma + \bar{a}\#\sigma + \alpha \# s,$$

and so for $0 < T < T(\varepsilon)$, $||A_0||_{H^1} + ||\bar{a}||_{L^\infty(H^1)} < \varepsilon$ and in view of our estimate for σ, we estimate

$$\left|\left|\frac{d}{dt}\alpha\right|\right|_{L^2} + ||\alpha||_{L^\infty(H^1)} + ||\alpha||_{L^2(H^2)} \leq C\varepsilon\left(\left|\left|\frac{d}{dt}\alpha\right|\right|_{L^2} + ||\alpha||_{L^\infty(H^1)} + ||\alpha||_{L^2(H^2)}\right).$$

Thus, for $\varepsilon > 0$ sufficiently small, we get $\alpha = 0$ and $\sigma = 0$, as claimed. □

From Propositions 5.6.1 and 5.5.2 the uniqueness of the local solution $D = D_{\bar{a}}$ to the YMHF as constructed in Section 5.5 follows.

5.7 Long Time Existence

In this section we will prove the claim about the maximality about the short time existence time, T and also briefly discuss the application of Uhlenbeck's removable singularities theorem to the long time existence of the flow, as analysed by Schlatter.

Proof of Theorem 5.2.3 (ii). Suppose that $T < \infty$ is maximal such that the YMHF has a weak solution D, which is gauge-equivalent to a smooth solution $\hat{D} = (\hat{S}_0^{-1})^*(D)$ on $(0, T)$. Assume then by contradiction that there exists $R > 0$ such that (5.11) holds. Then we have by Lemma 5.3.6 that

$$\lim_{t \to T} \hat{D}(t) = \hat{D}(T)$$

exists in H^1.

Therefore, for $t_0 < T$ sufficiently close to T we have that the local solution $\hat{D}'$ to the YMHF with inital data $\hat{D}'(t_0)$ constructed in Section 5.4 extends to an interval $[t_0, t_1)$ where $t_1 > T$. By the uniqueness of weak solutions of YMHF and equivariance of YMHF under time-independent gauge transformations, we have that $D(t) = (\hat{S}_0)^*(\hat{D}'(t))$ on $[t_0, T)$. Therefore, we have that $(\hat{S}_0)^*(\hat{D}'(t))$ extends the solution $D(t)$ to the interval $[t_0, t_1)$, which contradicts the maximality of T. Therefore the maximal existence time is characterised by (5.1).

Note that by (5.9) and since $\mathcal{YM}(D_0) < \infty$, we have that the energy contained in any singularity is finite. Since

$$\limsup_{t \nearrow \bar{t}_1} \int_{B_R(\bar{x}_1^i)} |F_{D(t)}|^2 * (1) \geq \varepsilon_0 > 0$$

we have that the energy contained in a singularity is also non-zero. Therefore the number of singularities must be finite, otherwise the energy contained within the singularities would be infinite, which is a contradiction to the finite initial energy assertion. Therefore the energy concentrates in at most finitely mani points. □

A full analysis of the long term existence and the asymptotic behaviour of the solution to the YMHF can be found in [45], although we discuss here how removable singularities plays a crucial role in this theorem. The theorem of Uhlenbeck which was proven in Chapter 4 must first be slightly extended to be able to remove singularities where the connection has finite energy, but is not necessarily Yang–Mills. This was proven shortly after her original singularities theorem by Uhlenbeck in [55], Theorem 2.1, although a more modern proof can be found in [41].

With this, the flow can be 'patched over' at times where the curvature concentrates (at finitely many points) and the argument described above for the short-time existence and uniqueness of the flow can be applied. This can be continued (what turns out to be) finitely many times to obtain a limiting connection. For the original proof see [45], although a detailed exposition can also be found at [18], Chapter 6.

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